Vectors

We will be using vectors and matrices to store and manipulate data.

Definition: A **vector** $\vec{\mathbf{v}}$ is a column of numbers. Use bold faced letters or vector signs to distinguish vectors from other variables.

We refer to the **entries** of a vector by using subscripts.

The **length** of a vector is the number of entries it has. (normally n)

Example. $$\vec{\mathbf{v}} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T.$$

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Example. Use a vector to represent the age distribution of a population: let F_i be the number of females with ages in the interval $\big[5i, 5(i+1)\big)$. We can represent the total female population by the vector $\vec{\mathbf{F}}$.
The females from 0 up to 5 are counted in F_0;
those from 5 up to 10 are counted in F_1, etc.

$$\vec{\mathbf{F}} = \begin{bmatrix} F_0 \\ F_1 \\ F_2 \\ \vdots \\ F_{n-1} \end{bmatrix}$$

Matrices

Definition: A **matrix** A is a two-dimensional array of numbers.

A matrix with m rows and n columns is called an $\overset{\text{“}m\text{ by }n\text{”}}{m \times n}$ matrix.

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Note: A vector can be thought of as an $n \times 1$ matrix.

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Matrices are denoted by a capital letter. Entries are lower case and have two subscripts, the corresponding row and column.

Example. A generic 2×3 matrix has the form $A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{bmatrix}$.

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Definition: The matrix $B = \begin{bmatrix} 30 & 50 \\ 100 & 250 \end{bmatrix}$ is a **square matrix** because it has the same number of rows as columns.

Matrices

Example. We will sometimes interpret a matrix as a **transition** matrix.

In this case, the matrix is square (say $n \times n$), where the n rows and n columns correspond to certain **states** (situations).

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Example. In our population example, suppose we want to model people getting older, transitioning from one state (age group) to the next. We would set up a transition matrix such as:

FROM state:

TO state:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

because everyone in the first age group will move to the second age group ($a_{2,1}$), everyone in state 2 will move to state 3 ($a_{3,2}$), etc.

Matrix Multiplication

The power of matrices arises in their multiplication.

Given two matrices, A of size $m \times k$ and B of size $l \times n$, we can find the product AB **if and only if** k equals l.

Let A be an $m \times k$ matrix and B, $k \times n$. Then AB is of size $m \times n$.

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To calculate the entries of AB, remember: "Row by column":

$$\begin{bmatrix} 1 & 4 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 6 \\ -4 & 1 & 2 \end{bmatrix} = \begin{bmatrix} \bigcirc & \bigcirc & \bigcirc \\ \bigcirc & \bigcirc & \bigcirc \end{bmatrix}$$

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When we write A^2, this means AA; A^3 means AAA, etc.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \bigcirc & \bigcirc \\ 0 & 1 & \bigcirc \\ 0 & 0 & 1 \end{bmatrix}$$

The power of transition matrices

Example. Modeling a changing population using a matrix model.

Let us choose a size of age interval Δ=5 years ("Delta"), and divide the female population into states:

State 0: ages $[0, 5)$ with $F_0 = 150$ females
State 1: ages $[5, 10)$ with $F_1 = 200$ females
State 2: ages $[10, 15)$ with $F_2 = 180$ females
State 3: ages $[15, 20)$ with $F_3 = 120$ females
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Using a transition matrix, we can determine the population in 5 years:

$$A \cdot \vec{\mathbf{F}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}^1 \begin{bmatrix} 150 \\ 200 \\ 180 \\ 120 \\ 60 \end{bmatrix} = \begin{bmatrix} 0 \\ 150 \\ 200 \\ 180 \\ 120 \end{bmatrix}$$

Leslie Matrices

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The resulting transition matrix is called a **Leslie matrix**:
Let m_i be the average number of females that women in state i bear.
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then $$\begin{bmatrix} F_0(t+\Delta) \\ F_1(t+\Delta) \\ F_2(t+\Delta) \\ \vdots \\ F_{n-1}(t+\Delta) \end{bmatrix} = \begin{bmatrix} m_0 & m_1 & m_2 & \cdots & m_{n-1} \\ p_0 & 0 & 0 & \cdots & 0 \\ 0 & p_1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & 0 & p_{n-2} & 0 \end{bmatrix} \begin{bmatrix} F_0(t) \\ F_1(t) \\ F_2(t) \\ \vdots \\ F_{n-1}(t) \end{bmatrix}$$

$$\vec{\mathbf{F}}(t+\Delta) = M \cdot \vec{\mathbf{F}}(t)$$

Leslie Matrices

Example. An animal population example (p. 47)
The population in three age groups, $F_0 = 80$, $F_1 = 40$, and $F_2 = 20$.

Suppose that as Δ time passes, everyone in state 2 dies, and one quarter of everyone else dies. Also suppose that the age-specific maternity rates are $m_0 = 0$, $m_1 = 1$, and $m_2 = 2$. Determine the Leslie matrix and the population distributions at times Δ and 2Δ.

$$\begin{bmatrix} & & \\ & 0 & 0 \\ 0 & & 0 \end{bmatrix} \begin{bmatrix} 80 \\ 40 \\ 20 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix} = \vec{\mathbf{F}}(\Delta)$$

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Example. Problem 1.5.6 from page 51.

(a) For the Leslie matrix $M = \begin{bmatrix} 3/2 & 2 \\ 1/2 & 0 \end{bmatrix}$, show that

$$M \begin{bmatrix} 4 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 4 \\ 1 \end{bmatrix} \text{ and } M \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

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- A Leslie matrix model is more descriptively realistic than the exponential model from Section 1.4, yet gives the same results.
- We've just worked with eigenvalues and eigenvectors!