Markov Chains

A **Markov chain** is a sequence of random variables from some sample space, each corresponding to a successive time interval. From one time interval to the next, there is a *fixed* probability $a_{i,j}$ of transitioning from state j to state i. No transition depends on a past transition.

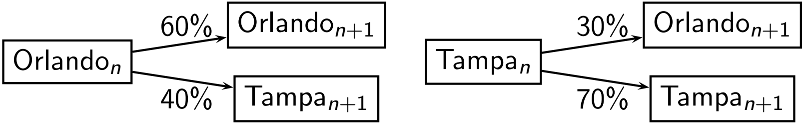
Keep track of these probabilities in an associated transition matrix A.

Markov Chains

A **Markov chain** is a sequence of random variables from some sample space, each corresponding to a successive time interval. From one time interval to the next, there is a *fixed* probability $a_{i,j}$ of transitioning from state j to state i. No transition depends on a past transition.

Keep track of these probabilities in an associated transition matrix A.

Example. Suppose you run a rental company based in Orlando and Tampa, Florida. People often drive between the cities; cars can be picked up and dropped off in either city. Suppose that historically,



What distribution of cars can the company expect in the long run?

Markov Chains

We will model this situation with a Markov Chain.

The historical data suggest that with a probability of 0.6, a car in Orlando at time n will be in Orlando at time $n+1$. Use this and the other expected transition probabilities to form the transition matrix A.

Markov Chains

We will model this situation with a Markov Chain.

The historical data suggest that with a probability of 0.6, a car in Orlando at time n will be in Orlando at time $n+1$. Use this and the other expected transition probabilities to form the transition matrix A.

FROM:

TO:

$$\begin{array}{c} \\ \text{Or} \\ \text{Tm} \end{array} \begin{array}{cc} \text{Or} & \text{Tm} \\ \left[\begin{array}{c} \quad \\ \quad \end{array} \right. & \left. \begin{array}{c} \quad \\ \quad \end{array} \right] \end{array} = A,$$

Markov Chains

We will model this situation with a Markov Chain.

The historical data suggest that with a probability of 0.6, a car in Orlando at time n will be in Orlando at time $n+1$. Use this and the other expected transition probabilities to form the transition matrix A.

FROM:

$$\textbf{TO}: \begin{array}{c} \\ \text{Or} \\ \text{Tm} \end{array} \begin{array}{c} \begin{array}{cc} \text{Or} & \text{Tm} \end{array} \\ \begin{bmatrix} 0.6 & \\ & \end{bmatrix} \end{array} = A,$$

Markov Chains

We will model this situation with a Markov Chain.

The historical data suggest that with a probability of 0.6, a car in Orlando at time n will be in Orlando at time $n+1$. Use this and the other expected transition probabilities to form the transition matrix A.

FROM:

$$\textbf{TO}: \begin{array}{c} \\ \text{Or} \\ \text{Tm} \end{array} \begin{array}{c} \begin{array}{cc} \text{Or} & \text{Tm} \end{array} \\ \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} \end{array} = A,$$

Markov Chains

We will model this situation with a Markov Chain.

The historical data suggest that with a probability of 0.6, a car in Orlando at time n will be in Orlando at time $n+1$. Use this and the other expected transition probabilities to form the transition matrix A.

FROM:

$$\textbf{TO}: \begin{array}{c} \\ \text{Or} \\ \text{Tm} \end{array} \begin{array}{c} \begin{array}{cc} \text{Or} & \text{Tm} \end{array} \\ \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} \end{array} = A,$$

- Let o_n be the probability that a car is in Orlando on day n
- Let t_n be the probability that a car is in Tampa on day n.

Markov Chains

We will model this situation with a Markov Chain.

The historical data suggest that with a probability of 0.6, a car in Orlando at time n will be in Orlando at time $n+1$. Use this and the other expected transition probabilities to form the transition matrix A.

FROM:

$$\textbf{TO}: \begin{array}{c} \\ \text{Or} \\ \text{Tm} \end{array} \begin{array}{c} \begin{array}{cc} \text{Or} & \text{Tm} \end{array} \\ \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} \end{array} = A,$$

- Let o_n be the probability that a car is in Orlando on day n
- Let t_n be the probability that a car is in Tampa on day n.

We can represent the distribution of cars at time n with the vector

$$\vec{\mathbf{x}}_n = \begin{bmatrix} o_n \\ t_n \end{bmatrix}.$$

Markov Chains

We will model this situation with a Markov Chain.

The historical data suggest that with a probability of 0.6, a car in Orlando at time n will be in Orlando at time $n+1$. Use this and the other expected transition probabilities to form the transition matrix A.

FROM:

$$\textbf{TO}: \begin{matrix} & \text{Or} & \text{Tm} \\ \text{Or} & \\ \text{Tm} & \end{matrix} \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} = A,$$

- Let o_n be the probability that a car is in Orlando on day n
- Let t_n be the probability that a car is in Tampa on day n.

We can represent the distribution of cars at time n with the vector $\vec{\mathbf{x}}_n = \begin{bmatrix} o_n \\ t_n \end{bmatrix}$. And so, $\vec{\mathbf{x}}_{n+1} = \begin{bmatrix} o_{n+1} \\ t_{n+1} \end{bmatrix} = A \cdot \begin{bmatrix} o_n \\ t_n \end{bmatrix} = A\vec{\mathbf{x}}_n$.

Markov Chains

We will model this situation with a Markov Chain.

The historical data suggest that with a probability of 0.6, a car in Orlando at time n will be in Orlando at time $n+1$. Use this and the other expected transition probabilities to form the transition matrix A.

FROM:

$$\textbf{TO}: \begin{array}{c} \\ \text{Or} \\ \text{Tm} \end{array} \begin{array}{c} \begin{array}{cc} \text{Or} & \text{Tm} \end{array} \\ \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} \end{array} = A,$$

- Let o_n be the probability that a car is in Orlando on day n
- Let t_n be the probability that a car is in Tampa on day n.

We can represent the distribution of cars at time n with the vector $\vec{\mathbf{x}}_n = \begin{bmatrix} o_n \\ t_n \end{bmatrix}$. And so, $\vec{\mathbf{x}}_{n+1} = \begin{bmatrix} o_{n+1} \\ t_{n+1} \end{bmatrix} = A \cdot \begin{bmatrix} o_n \\ t_n \end{bmatrix} = A\vec{\mathbf{x}}_n$.

Given an initial distribution $\vec{\mathbf{x}}_0 = \begin{bmatrix} o_0 \\ t_0 \end{bmatrix}$,

the expected distribution of cars at time n is $\vec{\mathbf{x}}_n =$ ____.

Markov Chains

For example, if they company starts off with twice as many cars in Orlando as in Tampa, then $\vec{\mathbf{x}}_0 = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$, so we expect

$$\vec{\mathbf{x}}_1 = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}.$$

Markov Chains

For example, if they company starts off with twice as many cars in Orlando as in Tampa, then $\vec{\mathbf{x}}_0 = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$, so we expect

$$\vec{\mathbf{x}}_1 = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}.$$

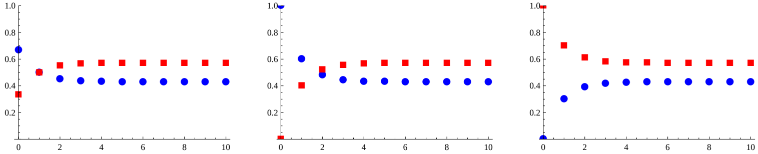
$$\vec{\mathbf{x}}_2 = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} \begin{bmatrix} \quad \\ \quad \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}.$$

Markov Chains

For example, if they company starts off with twice as many cars in Orlando as in Tampa, then $\vec{\mathbf{x}}_0 = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$, so we expect

$$\vec{\mathbf{x}}_1 = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}.$$

$$\vec{\mathbf{x}}_2 = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} \begin{bmatrix} \quad \\ \quad \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}.$$

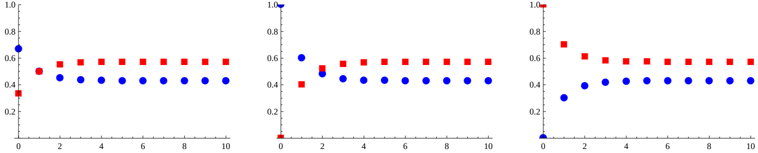


Markov Chains

For example, if they company starts off with twice as many cars in Orlando as in Tampa, then $\vec{\mathbf{x}}_0 = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$, so we expect

$$\vec{\mathbf{x}}_1 = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}.$$

$$\vec{\mathbf{x}}_2 = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} \begin{bmatrix} \quad \\ \quad \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}.$$



How do we determine the expected distribution in the long run?

Markov Chains

Definition: Given a Markov Chain with transition matrix A, an **equilibrium distribution** is a vector $\vec{\mathbf{x}}_{eq}$ that satisfies $A\vec{\mathbf{x}}_{eq} = \vec{\mathbf{x}}_{eq}$.

Markov Chains

Definition: Given a Markov Chain with transition matrix A, an **equilibrium distribution** is a vector $\vec{\mathbf{x}}_{eq}$ that satisfies $A\vec{\mathbf{x}}_{eq} = \vec{\mathbf{x}}_{eq}$.
[Linear Algebra: $\vec{\mathbf{x}}_{eq}$ *is an eigenvector corresponding to* $\lambda = 1$.]

Markov Chains

Definition: Given a Markov Chain with transition matrix A, an **equilibrium distribution** is a vector $\vec{\mathbf{x}}_{eq}$ that satisfies $A\vec{\mathbf{x}}_{eq} = \vec{\mathbf{x}}_{eq}$.

[Linear Algebra: $\vec{\mathbf{x}}_{eq}$ *is an eigenvector corresponding to* $\lambda = 1$.]

In our example, the equilibrium distribution satisfies

$$\begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} \begin{bmatrix} o_{eq} \\ t_{eq} \end{bmatrix} = \begin{bmatrix} o_{eq} \\ t_{eq} \end{bmatrix}.$$

Markov Chains

Definition: Given a Markov Chain with transition matrix A, an **equilibrium distribution** is a vector $\vec{\mathbf{x}}_{eq}$ that satisfies $A\vec{\mathbf{x}}_{eq} = \vec{\mathbf{x}}_{eq}$.
[Linear Algebra: $\vec{\mathbf{x}}_{eq}$ *is an eigenvector corresponding to* $\lambda = 1$.]

In our example, the equilibrium distribution satisfies

$$\begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} \begin{bmatrix} o_{eq} \\ t_{eq} \end{bmatrix} = \begin{bmatrix} o_{eq} \\ t_{eq} \end{bmatrix}.$$

So solve: $0.6o_{eq} + 0.3t_{eq} = o_{eq}$ and $0.4o_{eq} + 0.7t_{eq} = t_{eq}$.
Both equations reduce to $0.3t_{eq} = 0.4o_{eq}$, so $o_{eq} = \frac{3}{4}t_{eq}$.

Markov Chains

Definition: Given a Markov Chain with transition matrix A, an **equilibrium distribution** is a vector $\vec{\mathbf{x}}_{eq}$ that satisfies $A\vec{\mathbf{x}}_{eq} = \vec{\mathbf{x}}_{eq}$.

[Linear Algebra: $\vec{\mathbf{x}}_{eq}$ *is an eigenvector corresponding to* $\lambda = 1$.]

In our example, the equilibrium distribution satisfies

$$\begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} \begin{bmatrix} o_{eq} \\ t_{eq} \end{bmatrix} = \begin{bmatrix} o_{eq} \\ t_{eq} \end{bmatrix}.$$

So solve: $0.6o_{eq} + 0.3t_{eq} = o_{eq}$ and $0.4o_{eq} + 0.7t_{eq} = t_{eq}$.
Both equations reduce to $0.3t_{eq} = 0.4o_{eq}$, so $o_{eq} = \frac{3}{4}t_{eq}$.

Conclusion: If the company has 7000 cars in all, they would expect that in the long run, ____________________

Markov Chains

Definition: Given a Markov Chain with transition matrix A, an **equilibrium distribution** is a vector $\vec{\mathbf{x}}_{eq}$ that satisfies $A\vec{\mathbf{x}}_{eq} = \vec{\mathbf{x}}_{eq}$.
[Linear Algebra: $\vec{\mathbf{x}}_{eq}$ *is an eigenvector corresponding to* $\lambda = 1$.]

In our example, the equilibrium distribution satisfies

$$\begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} \begin{bmatrix} o_{eq} \\ t_{eq} \end{bmatrix} = \begin{bmatrix} o_{eq} \\ t_{eq} \end{bmatrix}.$$

So solve: $0.6o_{eq} + 0.3t_{eq} = o_{eq}$ and $0.4o_{eq} + 0.7t_{eq} = t_{eq}$.
Both equations reduce to $0.3t_{eq} = 0.4o_{eq}$, so $o_{eq} = \frac{3}{4}t_{eq}$.

Conclusion: If the company has 7000 cars in all, they would expect that in the long run, ______________________________

In Markov Chains: ★ The sum of the entries in every column of A is 1, because the total probability of transitioning **from** state i is 1.

Markov Chains

Definition: Given a Markov Chain with transition matrix A, an **equilibrium distribution** is a vector $\vec{\mathbf{x}}_{eq}$ that satisfies $A\vec{\mathbf{x}}_{eq} = \vec{\mathbf{x}}_{eq}$.
[Linear Algebra: $\vec{\mathbf{x}}_{eq}$ *is an eigenvector corresponding to* $\lambda = 1$.]

In our example, the equilibrium distribution satisfies

$$\begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} \begin{bmatrix} o_{eq} \\ t_{eq} \end{bmatrix} = \begin{bmatrix} o_{eq} \\ t_{eq} \end{bmatrix}.$$

So solve: $0.6o_{eq} + 0.3t_{eq} = o_{eq}$ and $0.4o_{eq} + 0.7t_{eq} = t_{eq}$.
Both equations reduce to $0.3t_{eq} = 0.4o_{eq}$, so $o_{eq} = \frac{3}{4}t_{eq}$.

Conclusion: If the company has 7000 cars in all, they would expect that in the long run, ________________________________

In Markov Chains: ★ The sum of the entries in every column of A is 1, because the total probability of transitioning **from** state i is 1.

★ There is no general rule for what the row sum will be.