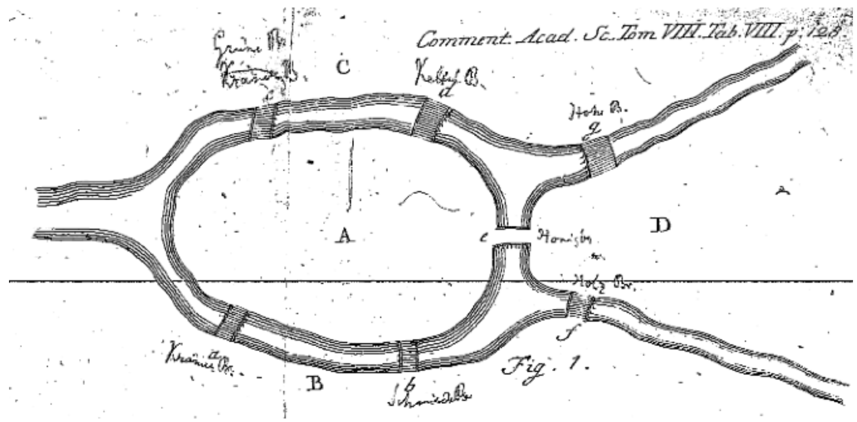
The Origins of Graph Theory

City of Königsberg in 1736



Question. Is it possible to start somewhere, cross all seven bridges exactly once, and return to where you started?

We can model this
with a graph:

Equiv. Question. Can we draw this graph without lifting our pencil?

Pseudographs

This is not a graph—it's a pseudograph.
For this section, we **allow** *multiple edges* and *loops*.

Types of "walks" in pseudographs:

Repeat Vertices?	Repeat Edges?	Open $A_1 \neq A_n$	Closed $A_1 = A_n$
No	No	path	cycle
Yes	No	trail	circuit
Yes	Yes	walk	closed walk

We need to update a few of our definitions.

Definition. The **length** of a "walk" is the number of *edges* involved.

Remark. In a simple graph, the smallest cycle possible is length 3.
In a pseudograph, there may exist cycles of length ______.

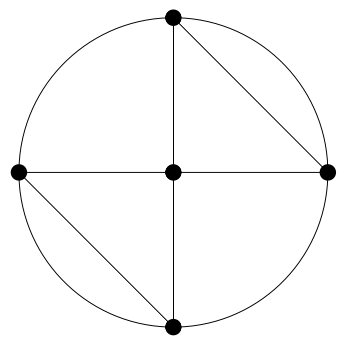
Definition. The **degree** of a vertex A is the number of edges incident with A; loops count twice!

Eulerian Circuits

Definitions.

An **Eulerian circuit** C in G is a circuit containing every edge of G.

An **Eulerian trail** T in G is a trail containing every edge of G.



T or **F**: A graph with an Eulerian circuit has an Eulerian trail.

The Königsberg bridge problem	$\longleftrightarrow$	Is there an Eulerian circuit in the corresponding pseudograph?

Characterization of Graphs with Eulerian Circuits

There is a simple way to determine if a graph has an Eulerian circuit.

Theorems 3.1.1 and 3.1.2. Let G be a pseudograph that is connected* *except possibly for isolated vertices*.

G has an Eulerian circuit $\iff$ the degree of every vertex is even.

Question. What about the Königsberg bridge pseudograph?

($\Rightarrow$) Euler, 1736. Given an Eulerian circuit C, each time a vertex appears in the circuit, there must be an "in edge" and an "out edge", so the total degree of each vertex must be even.

($\Leftarrow$) Hierholzer, 1873. This is harder; we need the following lemma.

Proof of Lemma 3.1.3

Lemma 3.1.3. If the degree of every vertex in a pseudograph is even, then every non-isolated vertex lies in some circuit in G.

Proof. Build a trail starting at any non-isolated vertex A in G.

When the trail arrives at a vertex B, what can we say about the number of edges incident to B **not yet traversed** by the trail?

So there is some edge to follow out of B; take it.

The trail must eventually return to A, giving us a circuit.

Proof of Theorem 3.1.2

Every vertex in G has even degree $\Rightarrow$ G has an Eulerian circuit

Find the longest circuit C in G. If C uses every edge, we are done.

Otherwise, it doesn't; we will aim to contradict the maximality of C:

Create H from G by deleting all edges of C & any isolated vertices.

Then H is a pseudograph where ______________________.

C and H must share a vertex A because ____________

Write C as $C = \cdots e_1 A e_2 \cdots$.

Find a circuit D in H through A. (Why?)

Write D as $D = \cdots f_1 A f_2 \cdots$. No edges of D repeat nor are they in C.

Define a new circuit $C' = \cdots e_1 A f_2 \cdots f_1 A e_2 \cdots$.

C' is a longer circuit in G than C, contradicting C's maximality. $\square$

Other related theorems

Theorem 3.1.6. Let G be a connected* pseudograph. Then,

G has an Eulerian trail $\Leftrightarrow$ G has exactly two vertices of odd degree.

Proof. Let x and y be the two vertices of odd degree.

Add edge xy to G.
Now $G + xy$ is a pseudograph ____________________.

By Theorem 3.1.2, there exists an Eulerian circuit in $G + xy$.

Remove xy from the circuit and you have an Eulerian trail in G.

Consequence. When drawing a picture without lifting your pencil, start and end at the vertices of odd degree!