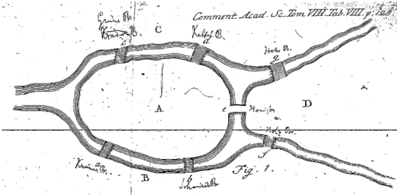
The Origins of Graph Theory

City of Königsberg in 1736



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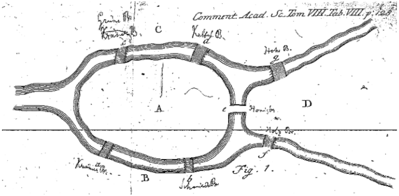

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We can model this
with a graph:

Equiv. Question. Can we draw this graph without lifting our pencil?

Pseudographs

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Definition. The **degree** of a vertex A is the number of edges incident with A; loops count twice!

Eulerian Circuits

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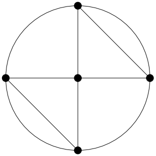

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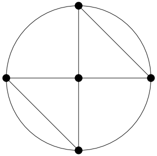

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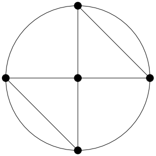

T or **F**: A graph with an Eulerian circuit has an Eulerian trail.

The Königsberg bridge problem	$\longleftrightarrow$	Is there an Eulerian circuit in the corresponding pseudograph?

Characterization of Graphs with Eulerian Circuits

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($\Leftarrow$) Hierholzer, 1873. This is harder; we need the following lemma.

Proof of Lemma 3.1.3

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The trail must eventually return to A, giving us a circuit.

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Consequence. When drawing a picture without lifting your pencil, start and end at the vertices of odd degree!