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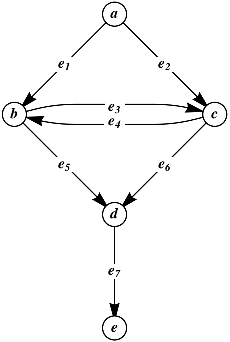

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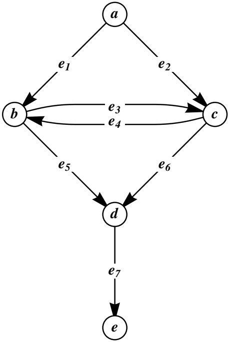

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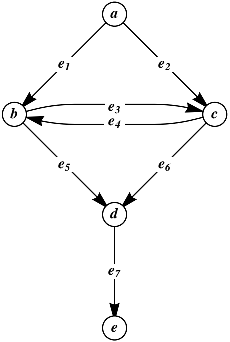

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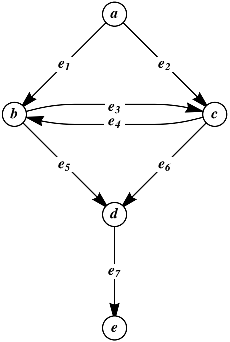

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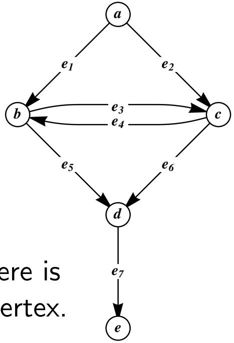
Definition. A digraph is **strongly connected** if there is a directed path from every vertex to every other vertex.



Generalizations to directed graphs

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We can generalize Theorems 3.1.1 and 3.1.2 to directed pseudographs:

Let G be a directed pseudograph that is strongly connected*.

G has an Eulerian circuit

$$\Updownarrow$$

the in-degree of every vertex equals
the out-degree of every vertex.

Application: de Bruijn sequences

Definition. An **alphabet** is a set $\mathcal{A} = \{a_1, \ldots, a_k\}$.
Definition. A **sequence** or **word** from $\mathcal{A}$ is a succession
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Theorem. A de Bruijn sequence of order n on $\mathcal{A}$ always exists.

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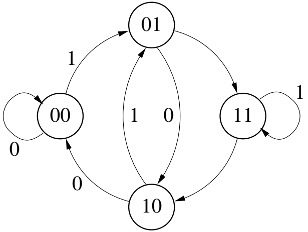

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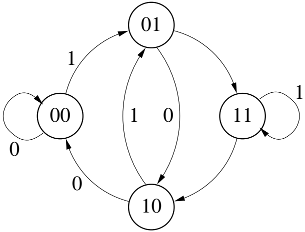
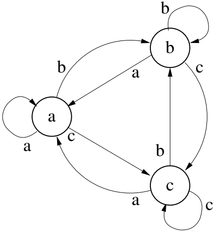


The de Bruijn graph of order 2 on the alphabet $\mathcal{A} = \{a, b, c\}$.



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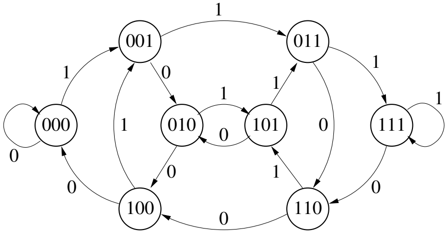
This is a different word for every edge! So every word appears in S.

Example: The binary de Bruijn graph of order 4



1. Find an Eulerian circuit in this graph.
2. Write down the corresponding sequence.
3. Verify that it is a de Bruijn sequence. (use chart, p.63)
4. Convince yourself that the name of a vertex is the same as the sequence formed by the three previous edges.