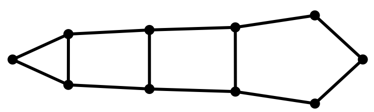
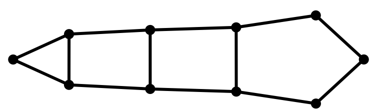
# Matchings in Graphs

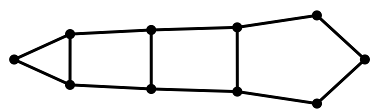
*Definition.* A **matching** $M$ in a graph $G$ is a subset of edges of $G$ that share no vertices.

*Definition.* A **maximal matching** $M$ is a matching such that the inclusion into $M$ of any edge of $G \setminus M$ is no longer a matching.

*Definition.* A **maximum matching** is a matching $M$ that has the most edges possible for the graph $G$.







*Definition.* A **perfect matching** is a matching involving every vertex.

*Thought Exercise:* What is the result of overlapping two matchings?

# Minimum vs. Minimal

We have just had two definitions related to the concept of "largest". An important concept is the distinction between

**maximum** and **maximal**.

**Maximum** refers to an element of absolute largest size.
(of *ALL* elts with *property*, this is largest.)

**Maximal** refers to an element of relative largest size.
(for *THIS* elt with *property*, no subset has property.)

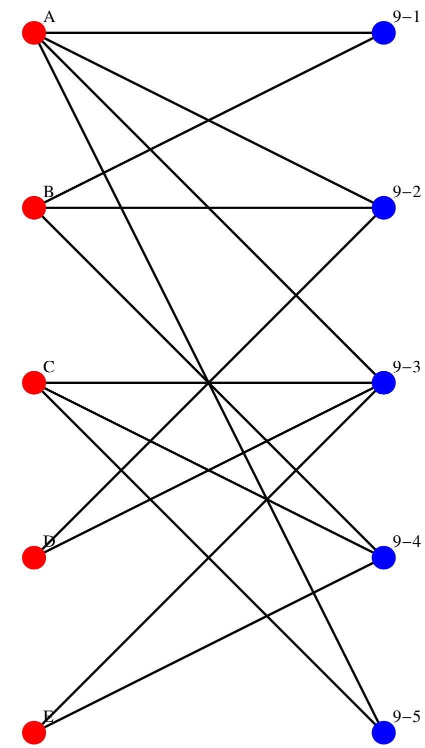
*Example.* maximal vs. maximal path in a graph:

*Example.* maximum vs. maximal clique in a graph:

# Application: Scheduling

Suppose you are working in a group trying to complete all the problems on the homework. Depending on everyone's preferences, you would like to assign each member one problem to do.

Person A likes problems 1, 2, 3, and 5.
Person B likes problems 1, 2, and 4.
Person C likes problems 3, 4, and 5.
Person D likes problems 2 and 3.
Person E likes problems 3 and 4.




Create a graph that models the situation.

*Question.*
What is a maximum matching for this graph?

We will use an algorithm to answer this question.

# Algorithms

*Definition.* An **algorithm** is a set of rules followed to solve a problem.

In general, an algorithm has the steps:

Havel–Hakimi:

1. Organize the input.

2. Repeatedly apply some steps
until a termination condition holds

3. Analyze data upon termination

Computers can be used to run the algorithms once we verify they work.

To verify the **correctness** of an algorithm:

1. Verify that the algorithm terminates. (often invoking finiteness)

2. Verify that the result satisfies the desired conditions.

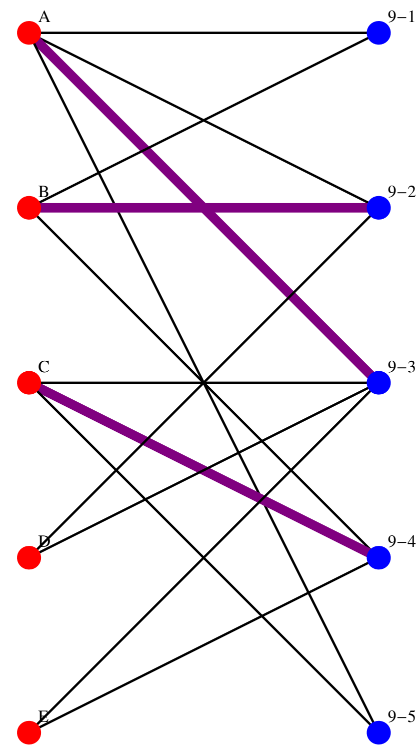
# Motivating The Hungarian Algorithm

Let us work through the basic idea behind the algorithm.
We start with an initial matching; we might as well make it maximal.
Why is the pictured matching maximal?

*Definition.* Given a matching $M$ in a graph $G$, an $M$-**alternating path** is a path in $G$ that starts at a vertex not in $M$, and whose edges alternate between being in $M$ and not in $M$.

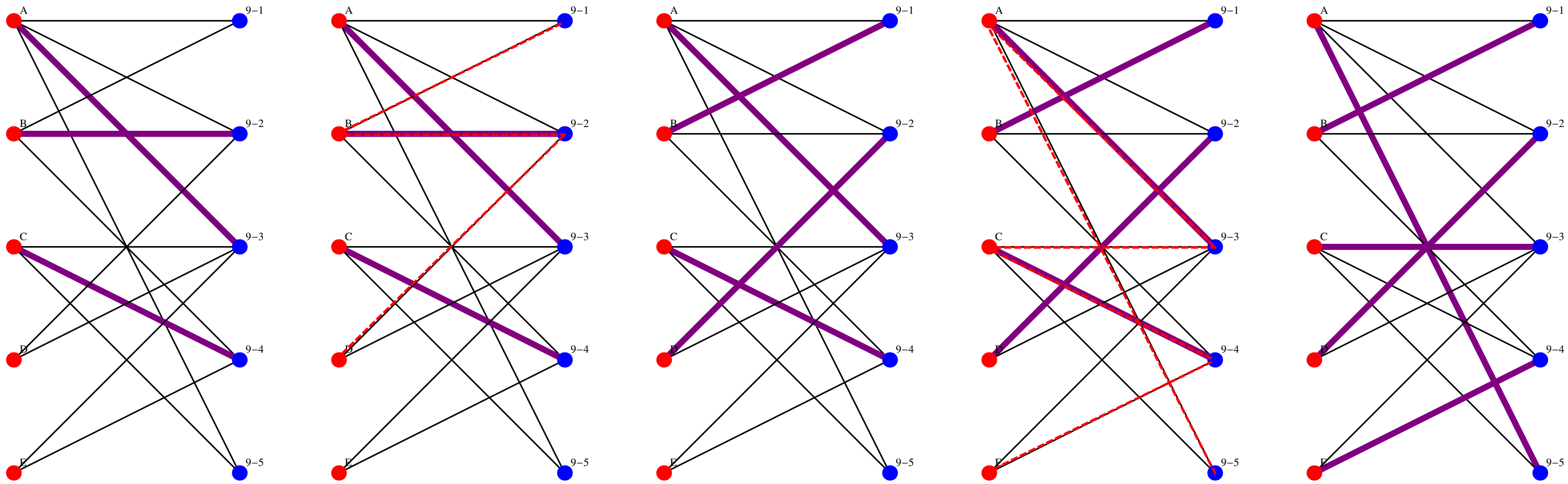
*Example.* $D \to 2 \to B \to 4 \to C$
is an $M$-alternating path.

*Definition.* An $M$-**augmenting path** is an $M$-alternating path that begins AND ends at unmatched vertices.




It is augmenting because we can improve $M$ by toggling the edges between those in $M$ and those not in $M$.

# Motivating The Hungarian Algorithm




Given $M$, $P = D \to 2 \to B \to 1$ is an $M$-augmenting path. Toggling the edges in $P$ gives a new matching $M'$.

Given $M'$, $P' = E \to 4 \to C \to 3 \to A \to 5$ is an $M'$-augmenting path. Toggling the edges in $P'$ gives a new matching $M''$.

The matching $M''$ is maximal. (Why?)

# The Hungarian Algorithm

*The Hungarian Algorithm* (Kuhn, Kőnig, Egeváry) [*Finds a maximum matching in a bipartite graph (w/red and blue vertices)*]

1. Start with a bipartite graph $G$ and any matching $M$. Label all red vertices *eligible* (for augmentation).
2. If all red, eligible vertices are matched, stop. Otherwise, there exists a red, unmatched, eligible vertex to use in the next step.

3. Let $v$ be an unmatched, eligible, red vertex. Start growing all possible $M$-alternating paths from $v$. That is, follow every edge not in $M$ to a blue vertex. From a matched blue vertex, follow the edge of $M$ back to a red vertex, and repeat as far as possible.

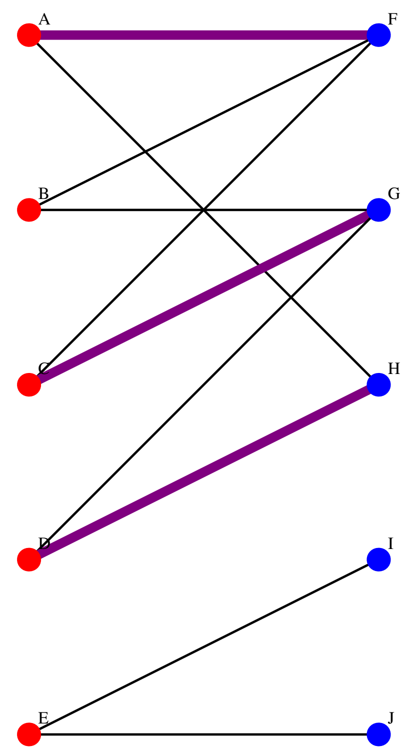
$$\begin{cases} \text{If there is an } M\text{-augmenting path, toggle edges to augment } M. \\ \text{If there is no } M\text{-augmenting path, mark } a \text{ ineligible.} \end{cases}$$

Return to Step 2.

# Applying the Hungarian Algorithm

Here is something that might happen during an application of the Hungarian algorithm:

*Example.* There is no $M$-augmenting path starting at $B$ in the graph to the right.




We would mark $B$ ineligible and move on to the next eligible, unmatched red vertex in the graph ($E$).

## Proof of Correctness

*Claim.* The Hungarian Algorithm gives a maximum matching.

*Proof.* We must show that the algorithm always stops, and that when it stops, the output is indeed a maximum matching.

**The algorithm terminates.** Each time Step 3 is run, one red vertex either becomes matched or becomes ineligible. Also, no red vertex that starts matched becomes unmatched. Since there are a finite number of red vertices, the algorithm must terminate.

**The output is a maximum matching.** The output $M$ is a matching inducing no $M$-augmenting paths in the graph. Suppose that there were another matching $M^*$ that used more edges than $M$.

When we overlap $M$ and $M^*$, the result is a union of cycles and paths. At least one path must have more edges from $M^*$ than $M$.

This path is an $M$-augmenting path, contradicting the definition of $M$.