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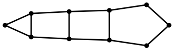

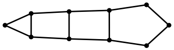

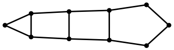

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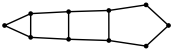
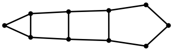
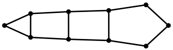

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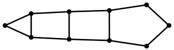
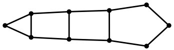
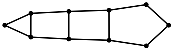

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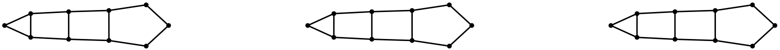

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Thought Exercise: What is the result of overlapping two matchings?

Minimum vs. Minimal

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Example. maximum vs. maximal clique in a graph:

Application: Scheduling

Suppose you are working in a group trying to complete all the problems on the homework. Depending on everyone's preferences, you would like to assign each member one problem to do.

Person A likes problems 1, 2, 3, and 5.
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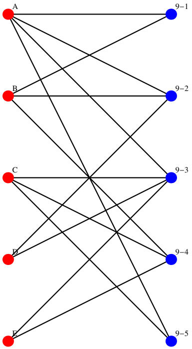

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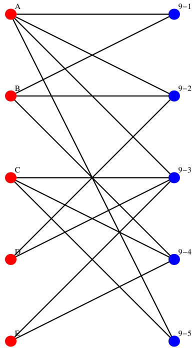

Create a graph that models the situation.

Question.
What is a maximum matching for this graph?

We will use an algorithm to answer this question.

Algorithms

Definition. An **algorithm** is a set of rules followed to solve a problem.

In general, an algorithm has the steps:

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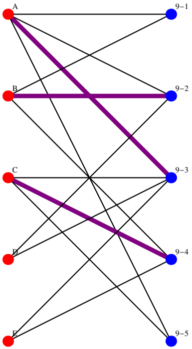
To verify the **correctness** of an algorithm:

1. Verify that the algorithm terminates. (often invoking finiteness)
2. Verify that the result satisfies the desired conditions.

Motivating The Hungarian Algorithm

Let us work through the basic idea behind the algorithm.
We start with an initial matching; we might as well make it maximal.
Why is the pictured matching maximal?



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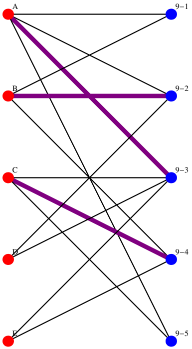

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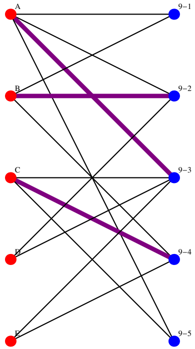

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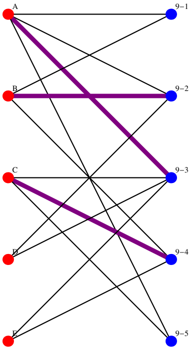
Definition. An M-**augmenting path** is an M-alternating path that begins AND ends at unmatched vertices.



It is augmenting because we can improve M by toggling the edges between those in M and those not in M.

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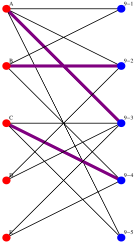

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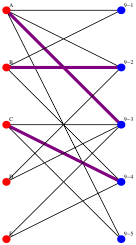

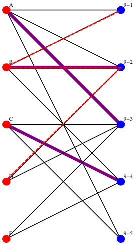

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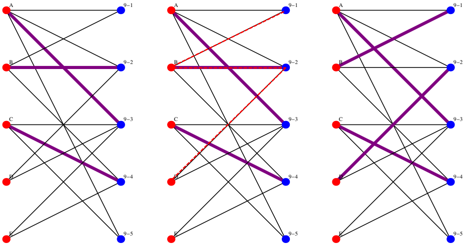

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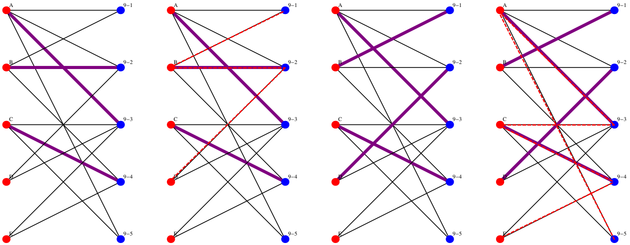

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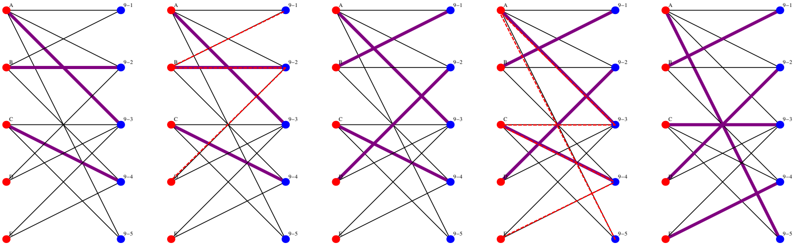

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The matching M'' is maximal. (Why?)

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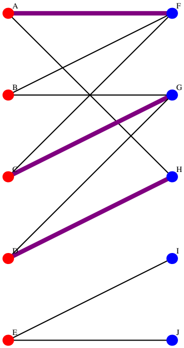
Return to Step 2.

Applying the Hungarian Algorithm

Here is something that might happen during an application of the Hungarian algorithm:

Example. There is no M-augmenting path starting at B in the graph to the right.



We would mark B ineligible and move on to the next eligible, unmatched red vertex in the graph (E).

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Claim. The Hungarian Algorithm gives a maximum matching.

Proof. We must show that the algorithm always stops, and that when it stops, the output is indeed a maximum matching.

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This path is an M-augmenting path, contradicting the definition of M.