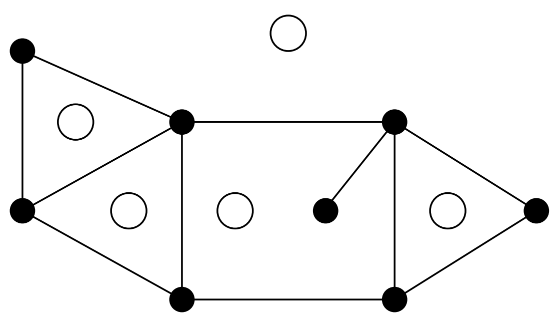
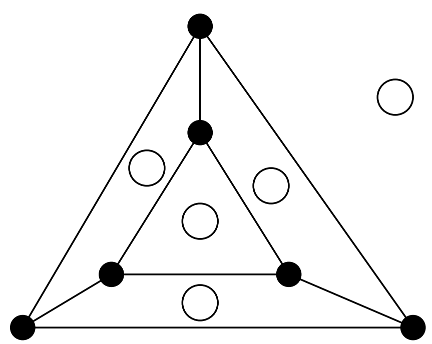
Dual Graphs

Definition. Given a plane drawing of a planar graph G, the **dual graph** $D(G)$ of G is a graph with vertices corresponding to the regions of G. Two vertices in $D(G)$ are connected by an edge each time the two regions share an edge as a border.



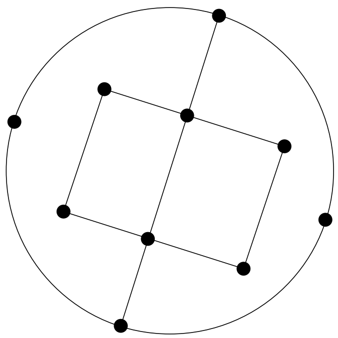


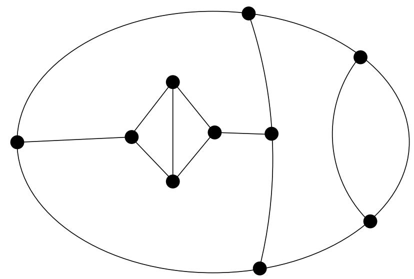
- The dual graph of a simple graph may not be simple.
  - Two regions may be adjacent multiple times.
- G and $D(G)$ have the same number of edges.

Definition. A graph G is **self-dual** if G is isomorphic to $D(G)$.

Maps

Definition. A *map* is a plane drawing of a connected, bridgeless, planar multigraph. If the map is 3-regular, then it is a **normal map**.





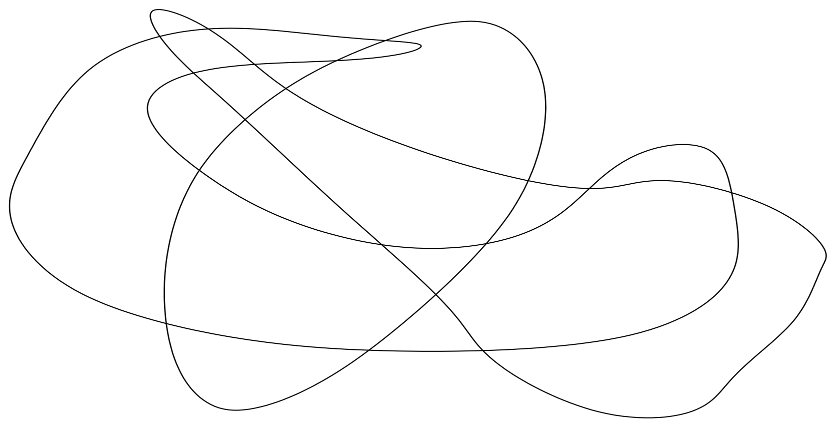
Definition. In a map, the regions are called **countries**. Countries may share several edges.

Definition. A **proper coloring** of a map is an assignment of colors to each country so that no two adjacent countries are the same color.

Question. How many colors are necessary to properly color a map?

Proper Map Colorings

Lemma 8.2.2. If M is a map that is a union of simple closed curves, the regions can be colored by two colors.



Proof. Color the regions R of M as follows:

$$\begin{Bmatrix} \text{red} & \text{if } R \text{ is enclosed in an odd number of curves} \\ \text{blue} & \text{if } R \text{ is enclosed in an even number of curves} \end{Bmatrix}.$$

This is a proper coloring of M. Any two adjacent regions are on opposite sides of a closed curve, so the number of curves in which each is enclosed is off by one.

The Four Color Theorem

Lemma 8.2.6. (The Four Color Theorem)
Every normal map has a proper coloring by four colors.

Proof. Very hard.

$\star$ This is the wrong object $\star$

Theorem. If G is a plane drawing of a maximal planar graph, then its dual graph $D(G)$ is a normal map.

Maximal Planar:

- Every face of G is a triangle $\rightsquigarrow$
- G is connected $\rightsquigarrow$
- G is planar $\rightsquigarrow$

The Four Color Theorem

Assuming Lemma 8.2.6,

$$\begin{aligned} G \text{ is maximal planar} \quad &\Rightarrow \quad D(G) \text{ is a normal map} \\ &\Rightarrow \quad \text{countries of } D(G) \text{ 4-colorable} \\ &\Rightarrow \quad \text{vertices of } G \text{ 4-colorable} \\ &\Rightarrow \quad \chi(G) \leq 4 \end{aligned}$$

This would prove:

Theorem 8.2.8. If G is maximal planar, then $\chi(G) \leq 4$.

Remember: Every planar graph is a subgraph of a maximal planar graph. So Lemma C implies:

Theorem 8.2.9. If G is a planar graph, then $\chi(G) \leq 4$.

$\star$ History $\star$