Course Notes

Combinatorics, Spring 2022

Queens College, Math 636

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http://qc.edu/~chanusa/courses/636/22/

What is combinatorics?

In this class: Learn how to count . . . **better**.

Question: How many domino tilings are there of an 8×8 chessboard?

A **domino tiling** is a placement of dominoes on a region, where

- Each domino covers two squares.
- The dominoes cover the whole region and do not overlap.

Domino tilings

How to determine the "answer"?

- Convert the chessboard into a combinatorial structure (a graph).
- Represent the graph numerically as a matrix.
- Take the determinant of this matrix.
- Use the structure of the matrices to determine their eigenvalues.

Question: How many domino tilings are there of an $m \times n$ board?

Answer: If m and n are both even, then we have the **formula** (!):

$$\prod_{j=1}^{m/2} \prod_{k=1}^{n/2} \left(4\cos^2 \frac{\pi j}{m+1} + 4\cos^2 \frac{\pi k}{n+1} \right).$$

Combinatorial questions

What kind of questions come up in combinatorics?

They are questions about discrete objects.

- Can we count the objects?
  - **Count** means give a *number*.
- Can we enumerate the arrangements?
  - **Enumerate** means give a *description* or *list*.
- Do any objects have a desired property?
  - This is an **existence** question.
- Can we construct an object with a desired property?
  - We need to find a method of **construction**.
- Is there a “best” object?
  - **Prove optimality**.

Mastering “Combinatorics” means internalizing techniques and strategies to know the best way to approach a counting question.

Uses a different kind of reasoning than in other math classes.

To do well in this class:

- **Come to class prepared.**
  - Print out and read over course notes.
  - Read sections before class.
- **Form good study groups.**
  - Discuss homework and classwork.
  - Bounce proof ideas around.
  - You will depend on this group.
- **Put in the time.**
  - Three credits = 6–9 hours per week out of class.
  - Homework stresses key concepts from class; learning takes time.
- **Stay in contact.**
  - If you are confused, ask questions (in class and out).
  - Don't fall behind in coursework or project.
  - I need to understand your concerns.

Visit the webpage. First homework (many parts!) due Wed.

Get to know each other

Arrange yourselves into groups.

- Introduce yourself. (your name, where you are from)
- What brought you to this class?
- Fill out **the front of** your notecard:
  - Write your name. (Stylize if you wish.)
  - Write some words about how I might remember you & your name.
  - *Draw* something (anything!) in the remaining space.
- Exchange contact information. (phone / email / other)
- *Small talk suggestion:* Did you do anything in the snow?

Four Counting Questions (p. 2)

Here are four counting questions.

Q1. How many 8-character passwords are there using A–Z, a–z, 0–9?

Q2. In how many ways can a baseball manager order nine fixed baseball players in a lineup?

Q3. How many Pick-6 lottery tickets are there? (Choose six numbers between 1–40.)

Q4. How many possible orders for a dozen donuts are there when the store has 30 varieties?

Group discussion: Use your powers of estimation to order these from smallest to largest.

$$\_\_\_\_\_ < \_\_\_\_\_ < \_\_\_\_\_ < \_\_\_\_\_$$

Counting words

Definition: A **list** or **word** is an ordered sequence of objects.

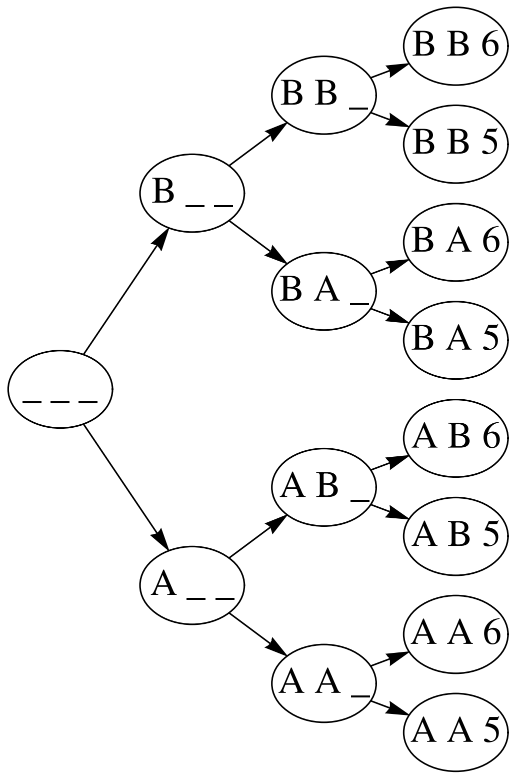
Definition: A k-**list** or k-**word** is a list of length k.

- A **list** or **word** is always ordered and a **set** is always unordered.

Question: How many lists have three entries where

- The first two entries can be either A or B.
- The last entry is either 5 or 6.

Answer: We can solve this using a tree diagram:



Alternatively: Notice two *independent* choices for each character. Multiply $2 \cdot 2 \cdot 2 = 8$.

The Product Principle

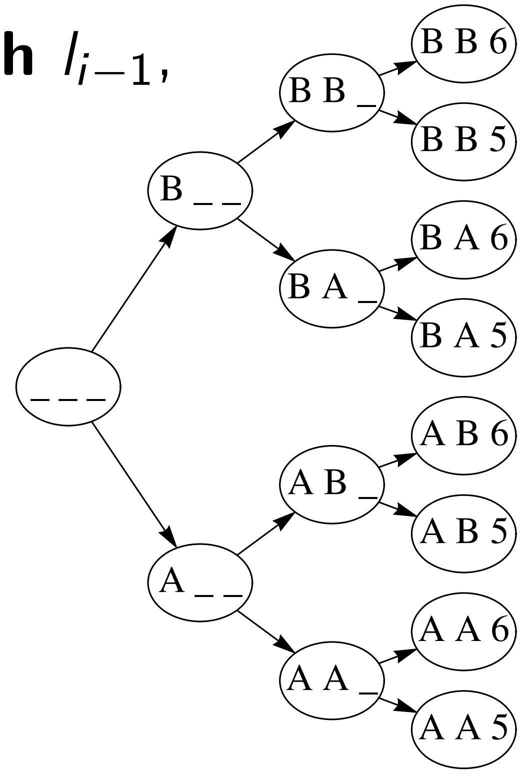
This illustrates:

The product principle: When counting lists $(l_1, l_2, \ldots, l_k)$,

IF there are c_1 choices for entry l_1, each leading to a different list,

AND IF there are c_i choices for entry l_i,
no matter the choices made for l_1 through l_{i-1},
each leading to a different list

THEN there are $c_1 c_2 \cdots c_k$ such lists.



Caution: The product principle seems simple,
but we must be careful when we use it.

Lists WITH repetition

Q1. How many 8-character passwords are there using A–Z, a–z, 0–9?

Answer: Creating a word of length 8, with ____ choices for each character. Therefore, the number of 8-character passwords is ____.
(=218,340,105,584,896)

In general, the number of words of length k that can be made from an alphabet of length n and <u>where repetition is allowed</u> is n^k

Application: Counting Subsets

Example. How many subsets of a set $S = \{s_1, s_2, \ldots, s_n\}$ are there?

Strategy: "Try small problems, see a pattern."

- $n = 0$: $S = \emptyset \rightsquigarrow \{\emptyset\}$, size 1.
- $n = 1$: $S = \{s_1\} \rightsquigarrow \{\emptyset, \{s_1\}\}$, size 2.
- $n = 2$: $S = \{s_1, s_2\} \rightsquigarrow \{\emptyset, \{s_1\}, \{s_2\}, \{s_1, s_2\}\}$, size 4.
- $n = 3$: $S = \{s_1, s_2, s_3\} \rightsquigarrow \left\{\begin{matrix} \emptyset, & \{s_1\}, & \{s_2\}, & \{s_1, s_2\}, \\ \{s_3\}, & \{s_1, s_3\}, & \{s_2, s_3\}, & \{s_1, s_2, s_3\} \end{matrix}\right\}$, 8.

It appears that the number of subsets of S is ______. (notation)

This number also counts ______________________________.

Equiv.: We can label the subsets by whether or not they contain s_i.

For example, for $n = 3$, we label the subsets $\left\{\begin{matrix} 000, 100, 010, 110, \\ 001, 101, 011, 111 \end{matrix}\right\}$.

Permutations

Q2. In how many ways can a baseball manager order nine fixed baseball players in a lineup?

Answer: The number of choices for each lineup spot are:

____ ____ ____ ____ ____ ____ ____ ____ ____.

Multiplying gives that the number of lineups is ____ $= 362{,}880$.

Definition: A **permutation** of an n-set S is an (ordered) list of **all** elements of S. There are $n!$ such permutations.

Definition: A k-**permutation** of an n-set S is an (ordered) list of k distinct elements of S.

- "Permutation" always refers to a list without repetition.

Question: How many k-permutations of n are there?

Lists WITHOUT repetition

Question: How many 8-character passwords are there using A–Z, a–z, 0–9, containing no repeated character?

OK: 2eas3FGS, 10293465 **Not OK:** 2kdjfng2, oOoOoOo0

Answer: The number of choices for each character are:

____ ____ ____ ____ ____ ____ ____ ____,

for a total of $(62)_8 = \frac{62!}{54!}$ passwords.

In general, the number of words of length k that can be made from an alphabet of length n and where repetition is NOT allowed is $(n)_k$.

- That is, the number of k-permutations of an n-set is $(n)_k$.
- Special case: For n-permutations of an n-set: $n!$.

Notation

Some quantities appear frequently, so we use shorthand notation:

- $[n] := \{1, 2, \ldots, n\}$
- $2^S :=$ set of all subsets of S
- $n! := n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$
- $(n)_k := n \cdot (n-1) \cdot (n-2) \cdots (n-k+1) = \dfrac{n!}{(n-k)!}$
- $\dbinom{n}{k} := \dfrac{n!}{k!(n-k)!} = \dfrac{(n)_k}{k!}$
- $\left(\!\!\dbinom{n}{k}\!\!\right) := \dbinom{k+n-1}{k}$

★ Leave answers to counting questions in terms of these quantities.

★ **Do NOT** multiply out unless you are comparing values.

Counting subsets of a set

My question: In how many ways are there to choose a subset of k objects out of a set of n objects?

Your answer: $\binom{n}{k}$. "n choose k".

Question: In how many ways can you choose 4 objects out of 10?

Q3. How many Pick-6 lottery tickets are there?
(Choose six numbers between 1–40.)

$$= 3{,}838{,}380.$$

- $\binom{n}{k}$ is called a **binomial coefficient**.
- Alternate phrasing: How many k-subsets of an n-set are there?
- The individual objects we are counting are unordered.
  They are subsets, not lists.

A formula for $\binom{n}{k}$

You may know that $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{1}{k!}(n)_k$. But why?

Let's rearrange it. And prove it!

$$(n)_k = \binom{n}{k} k!$$

We ask the question:

"In how many ways are there to create a k-list of an n-set?"

LHS:

RHS:

Since we counted the same quantity twice, they must be equal!

Counting Multisets

Definition: A **multiset** is an unordered collection of elements where repetition is allowed.

- Example. $\{a, a, b, d\}$ is a multiset.

Definition: We say M is a **multisubset** of a set (or multiset) S if every element of M is an element of S.

- Example. $M = \{a, a, a, b, d\}$ is a **multisubset** of $S = \{a, b, c, d\}$.

Think Write Pair Share: Enumerate **all** multisubsets of [3].

[In other words, *list them all* or *completely describe the list*.]

Answer:

How would you describe a k-multisubset of $[n]$?

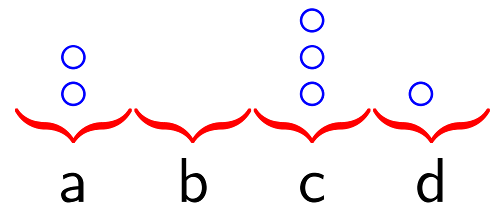
Balls and Walls

Question: How many k-multisets can be made from an n-set?

$\{a^2, b^0, c^3, d^1\}$ $n = 4$, $k = 6$

— is the same as —

Question: How many ways are there to place k indistinguishable balls into n distinguishable bins?



— is the same as —

Question: How many $\{\circ, |\}$-words contain k balls and $(n-1)$ walls?

$\circ\circ \mid \mid \circ\circ\circ \mid \circ$

— which we can count by: —

Question: How many ways to choose k ball pos'ns out of $k + n - 1$ total?

$\binom{k+n-1}{k} =: \left(\!\binom{n}{k}\!\right)$

Answering Q1–Q4

Q4. How many possible orders for a dozen donuts are there when the store has 30 varieties?

Answer: $\left(\!\!\binom{\ }{\ }\!\!\right) = \binom{\ }{\ } = 7{,}898{,}654{,}920.$

Correct order:

Q2.	Order 9 baseball players $(9!)$	362,880
Q3.	Pick-6; numbers 1–40 $\binom{40}{6}$	3,838,380
Q4.	12 donuts from 30 $\left(\!\!\binom{30}{12}\!\!\right)$	7,898,654,920
Q1.	8-character passwords (62^8)	218,340,105,584,896

Summary

	order matters (choose a list)	order doesn't matter (choose a set)
repetition allowed		
repetition not allowed		