Introduction to Bijections

Goal: Prove that two sets A and B are of the same size.

Tool: A **bijection** pairs up the elements of A and B.

Example. The set A of subsets of $\{s_1, s_2, s_3\}$ are in bijection with the set B of binary words of length 3.

Set A: $\{\ \emptyset,\ \{s_1\}, \{s_2\}, \{s_1, s_2\}, \{s_3\}, \{s_1, s_3\}, \{s_2, s_3\}, \{s_1, s_2, s_3\}\}$

Bijection:

Set B: $\{000, 100, 010, 110, 001, 101, 011, 111\}$

Rule: Given $a \in A$, (a is a subset), define $b \in B$ (b is a word):

Difficulties:

- Finding the rule (requires rearranging, ordering)
- Proving it is a bijection (requires logical reasoning).

What is a Function?

Reminder: A **function** f from A to B (write $f : A \to B$) is a rule where for each element $a \in A$, $f(a)$ is defined to be an element $b \in B$ (write $f : a \mapsto b$).

- f is **well-defined** if for all $a \in A$, $f(a) \in B$ and is unambiguous.

- A is called the **domain**. (We write $A = \text{dom}(f)$)
- B is called the **codomain**. (We write $B = \text{cod}(f)$)
- The **range** of f is the set of values that f takes on:

$$\text{rng}(f) = \left\{b \in B : f(a) = b \text{ for at least one } a \in A\right\}$$

Example. Let S be the set of 3-subsets of $[n]$ and let L be the set of 3-lists of $[n]$. Then define $f : S \to L$ to be the function that takes a 3-subset $\{i_1, i_2, i_3\} \in S$ (with $i_1 \leq i_2 \leq i_3$) to the list $(i_1, i_2, i_3) \in L$.

Question: Is f well-defined? Is $\text{rng}(f) = L$?

What is a Bijection?

Definition: A function $f : A \to B$ is **one-to-one** (an **injection**) when

For each $a_1, a_2 \in A$, if $f(a_1) = f(a_2)$, then $a_1 = a_2$.

Equivalently,

For each $a_1, a_2 \in A$, if $a_1 \neq a_2$, then $f(a_1) \neq f(a_2)$.

"When the inputs are different, the outputs are different." (picture)

Definition: A function $f : A \to B$ is **onto** (a **surjection**) when

For each $b \in B$, there exists some $a \in A$ such that $f(a) = b$.

"Every output gets hit."

Definition: A function $f : A \to B$ is a **bijection** if it is both one-to-one and onto.

The function from the previous page is ______________________.

Give an example of a function that is onto and not one-to-one.

Proving a Bijection

Example. Use a bijection to prove that $\binom{n}{k} = \binom{n}{n-k}$ for $0 \leq k \leq n$.

Proof. We first find two sets of those sizes:

Let A be the set of k-subsets of $[n]$ and (Size =)

Let B be the set of $(n-k)$-subsets of $[n]$. (Size =)

Step 1: Find a candidate bijection.

Strategy. Try out a small (enough) example. Try $n = 5$ and $k = 2$.

$$\left\{ \begin{array}{c} \{1,2\},\ \{1,3\} \\ \{1,4\},\ \{1,5\} \\ \{2,3\},\ \{2,4\} \\ \{2,5\},\ \{3,4\} \\ \{3,5\},\ \{4,5\} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \{1,2,3\},\ \{1,2,4\} \\ \{1,2,5\},\ \{1,3,4\} \\ \{1,3,5\},\ \{1,4,5\} \\ \{2,3,4\},\ \{2,3,5\} \\ \{2,4,5\},\ \{3,4,5\} \end{array} \right\}$$

Guess: Let S be a k-subset of $[n]$. Perhaps $f(S) = $ ______.

Proving a Bijection

Step 2: Prove f is well defined.

The function f is well defined. If S is any k-subset of $[n]$, then

Step 3: Prove f is a bijection.

Strategy. Prove that f is both one-to-one and onto.

f is 1-to-1:

f is onto:

We conclude that f is a bijection and therefore, $\binom{n}{k} = \binom{n}{n-k}$.

Alternative methods to prove bijections

Prove that a rule f is a bijection by finding f's **inverse**:

- Determine a rule for a candidate inverse function g.
- Show that f is a well defined function from A to B.
- Show that g is a well defined function from B to A.
- Show that f and g are two-sided inverses:
  Show for all $a \in A$, $g(f(a)) = a$
  and for all $b \in B$, $f(g(b)) = b$

Then both f and g are bijections.

Using the inverse function

Example. There exists as many even-sized subsets of $[n]$ as odd-sized subsets of $[n]$.

$$\text{even:} \quad \{\ \emptyset,\ \{s_1, s_2\}, \{s_1, s_3\},\ \{s_2, s_3\}\ \}$$
$$\text{odd:} \quad \{\{s_1\},\ \{s_2\},\ \{s_3\},\ \{s_1, s_2, s_3\}\}$$

Proof. Let A be the set of even-sized subsets of $[n]$ and let B be the set of odd-sized subsets of $[n]$. Consider the function

$$f(S) = \begin{cases} S \setminus \{1\} & \text{if } 1 \in S \\ S \cup \{1\} & \text{if } 1 \notin S \end{cases}.$$

- f is a well defined function from A to B (why?).
- f is also a well defined function from B to A (why?).
- f^2 is the identity function.

Therefore, f is a bijection, proving the statement, as desired.

Eyebrow-Raising Consequence: $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$

Pascal's triangle

Pascal's identity is the recurrence $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.
With initial conditions we can calculate $\binom{n}{k}$ for all n and k.
$\binom{n}{0} = 1$ and $\binom{n}{n} = 1$ for all n.

$n \backslash k$	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	2	1					
3	1	3	3	1				
4	1	4	6	4	1			
5	1	5	10	10	5	1		
6	1	6	15	20	15	6	1	
7	1							1

Seq's in Pascal's triangle:

$1, 2, 3, 4, 5, \ldots$	$\binom{n}{1}$
$(a_n = n)$	A000027
$1, 3, 6, 10, 15, \ldots$	$\binom{n}{2}$
triangular	A000217
$1, 4, 10, 20, 35, \ldots$	$\binom{n}{3}$
tetrahedral	A000292
$1, 2, 6, 20, 70, \ldots$	$\binom{2n}{n}$
centr. binom.	A000984

Online Encyclopedia of Integer Sequences:
`http://oeis.org/`

Binomial Theorem

Theorem 2.2.2. Let n be a positive integer. For all x and y,

$$(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \cdots + \binom{n}{n-1}xy^{n-1} + y^n.$$

In other words: The n-th row of Pascal's triangle contains the coefficients of the terms in the expansion of $(x+y)^n$.

Proof. In the expansion of $(x+y)(x+y)\cdots(x+y)$, in how many ways can a term have the form $x^{n-k}y^k$?

Question: What happens when $x = 1$ and $y = -1$?