Introduction to Bijections

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Set A: $\{\ \emptyset,\ \{s_1\}, \{s_2\}, \{s_1, s_2\}, \{s_3\}, \{s_1, s_3\}, \{s_2, s_3\}, \{s_1, s_2, s_3\}\}$

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Difficulties:

- Finding the rule (requires rearranging, ordering)
- Proving it is a bijection (requires logical reasoning).

What is a Function?

Reminder: A **function** f from A to B (write $f : A \to B$) is a rule where for each element $a \in A$, $f(a)$ is defined to be an element $b \in B$ (write $f : a \mapsto b$).

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Example. Let S be the set of 3-subsets of $[n]$ and let L be the set of 3-lists of $[n]$. Then define $f : S \to L$ to be the function that takes a 3-subset $\{i_1, i_2, i_3\} \in S$ (with $i_1 \le i_2 \le i_3$) to the list $(i_1, i_2, i_3) \in L$.

Question: Is f well-defined? Is $\text{rng}(f) = L$?

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Give an example of a function that is onto and not one-to-one.

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Step 1: Find a candidate bijection.

Strategy. Try out a small (enough) example. Try $n = 5$ and $k = 2$.

$$\left\{ \begin{array}{c} \{1,2\},\ \{1,3\} \\ \{1,4\},\ \{1,5\} \\ \{2,3\},\ \{2,4\} \\ \{2,5\},\ \{3,4\} \\ \{3,5\},\ \{4,5\} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \{1,2,3\},\ \{1,2,4\} \\ \{1,2,5\},\ \{1,3,4\} \\ \{1,3,5\},\ \{1,4,5\} \\ \{2,3,4\},\ \{2,3,5\} \\ \{2,4,5\},\ \{3,4,5\} \end{array} \right\}$$

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Guess: Let S be a k-subset of $[n]$. Perhaps $f(S) =$ ______.

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f is 1-to-1:

f is onto:

We conclude that f is a bijection and therefore, $\binom{n}{k} = \binom{n}{n-k}$.

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Then both f and g are bijections.

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Proof. Let A be the set of even-sized subsets of $[n]$ and let B be the set of odd-sized subsets of $[n]$. Consider the function

$$f(S) = \begin{cases} S \setminus \{1\} & \text{if } 1 \in S \\ S \cup \{1\} & \text{if } 1 \notin S \end{cases}.$$

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Therefore, f is a bijection, proving the statement, as desired.

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Eyebrow-Raising Consequence: $\displaystyle\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$

Pascal's triangle

Pascal's identity is the recurrence $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.
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$n \backslash k$	0	1	2	3	4	5	6	7
0	1							
1	1	1						
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Seq's in Pascal's triangle:

- $1, 2, 3, 4, 5, \ldots$ $\binom{n}{1}$
  ($a_n = n$)
- $1, 3, 6, 10, 15, \ldots$ $\binom{n}{2}$
  triangular
- $1, 4, 10, 20, 35, \ldots$ $\binom{n}{3}$
  tetrahedral
- $1, 2, 6, 20, 70, \ldots$ $\binom{2n}{n}$
  centr. binom.

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Online Encyclopedia of Integer Sequences:
http://oeis.org/

Binomial Theorem

Theorem 2.2.2. Let n be a positive integer. For all x and y,

$$(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \cdots + \binom{n}{n-1}xy^{n-1} + y^n.$$

In other words: The n-th row of Pascal's triangle contains the coefficients of the terms in the expansion of $(x+y)^n$.

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Question: What happens when $x = 1$ and $y = -1$?