

# Introduction to Symmetry

Many combinatorial objects have a natural symmetry.

**Example.** In how many ways can we seat 4 people at a round table?

There are  $4!$  permutations; however, each of \_\_\_\_\_ rotations gives the same order of guests. *Dividing* gives the \_\_\_\_\_ arrangements.

- ▶ In how many ways can we arrange 10 people into five pairs?
- ▶ In how many ways can we  $k$ -color the vertices of a square?

In order to approach counting questions involving symmetry rigorously, we use the mathematical notion of *equivalence relation*.

## Equivalence relations

*Definition:* An **equivalence relation**  $\mathcal{E}$  on a set  $A$  satisfies the following properties:

- ▶ **Reflexive:** For all  $a \in A$ ,  $a\mathcal{E}a$ .
- ▶ **Symmetric:** For all  $a, a' \in A$ , if  $a\mathcal{E}a'$ , then  $a'\mathcal{E}a$ .
- ▶ **Transitive:** For all  $a, a', a'' \in A$ , if  $a\mathcal{E}a'$ , and  $a'\mathcal{E}a''$ , then  $a\mathcal{E}a''$ .

*Example.* When sitting four people at a round table, let  $A$  be all 4-permutations. We say  $a = (a_1, a_2, a_3, a_4)$  and  $a' = (a'_1, a'_2, a'_3, a'_4)$  are “equivalent” ( $a\mathcal{E}a'$ ) if they are rotations of each other.

Verify that  $\mathcal{E}$  is an equivalence relation.

- ▶ It is reflexive because:
- ▶ It is symmetric because:
- ▶ It is transitive because:

# Equivalence classes

It is natural to investigate the set of all elements related to  $a$ :

*Definition:* The **equivalence class containing**  $a$  is the set

$$\mathcal{E}(a) = \{x \in A : x \mathcal{E} a\}.$$

**Class 1:**  $\{ (1,2,3,4) , (2,3,4,1) , (3,4,1,2) , (4,1,2,3) \}$

**Class 2:**  $\{ (1,2,4,3) , (2,4,3,1) , (4,3,1,2) , (3,1,2,4) \}$

**Class 3:**  $\{ (1,3,2,4) , (3,2,4,1) , (2,4,1,3) , (4,1,3,2) \}$

**Class 4:**  $\{ (1,3,4,2) , (3,4,2,1) , (4,2,1,3) , (2,1,3,4) \}$

**Class 5:**  $\{ (1,4,2,3) , (4,2,3,1) , (2,3,1,4) , (3,1,4,2) \}$

**Class 6:**  $\{ (1,4,3,2) , (4,3,2,1) , (3,2,1,4) , (2,1,4,3) \}$

- ▶ Our original question asks to count *equivalence classes (!)*.
- ▶ *Theorem 1.4.3.* If  $a \mathcal{E} a'$ , then  $\mathcal{E}(a) = \mathcal{E}(a')$ .
- ▶ Every element of  $A$  is in *one* and *only one* equivalence class.
  - ▶ We say: “The equivalence classes of  $\mathcal{E}$  partition  $A$ .”

## Equivalence classes partition $A$

**Definition:** A **partition** of a set  $S$  is a set of non-empty disjoint subsets of  $S$  whose union is  $S$ .

**Example.** Partitions of  $S = \{\circ, \heartsuit, \clubsuit, ?\}$  include:

- ▶  $\{\{\circ, \heartsuit\}, \{?\}, \{\clubsuit\}\}$
- ▶  $\{\{\heartsuit, \clubsuit\}, \{\circ, ?\}\}$

Every element is in some subset and no element is in multiple subsets.

**Key idea:** (Thm 1.4.5) The set of equivalence classes of  $A$  partitions  $A$ .

- ▶ Every equivalence class is non-empty.
- ▶ Every element of  $A$  is in *one* and *only one* equivalence class.

**The equivalence principle:** (p. 37) Let  $\mathcal{E}$  be an equivalence relation on a finite set  $A$ . If every equivalence class has size  $C$ , then  $\mathcal{E}$  has  $|A|/C$  equivalence classes. (DIVISION!)

## Permutations of multisets

**Example.** How many different orderings are there of the letters in the word MISSISSIPPI?

**Setup:** If the letters were all distinguishable, we would have a permutation of 11 letters,  $\{M, P, P, I, I, I, I, S, S, S, S\}$ , so  $|A| =$

Define  $a \mathcal{E} a'$  if  $a$  and  $a'$  are the same word when color is ignored. (Is this an equivalence relation?)

**Question:** How many words are in the same equivalence class?

Alternatively, count directly.

- ▶ In how many ways can you position the  $S$ 's?
- ▶ With  $S$ 's placed, how many choices for the  $I$ 's?
- ▶ With  $S$ 's,  $I$ 's placed, how many choices for the  $P$ 's?
- ▶ With  $S$ 's,  $I$ 's,  $P$ 's placed, how many choices for the  $M$ ?

## Words of caution

**Careful:** Conjugacy classes might not be of equal size.

**Example.** Let  $A$  be the subsets of  $[4]$ . Define  $S \mathcal{E} T$  when  $|S| = |T|$ . Determine the number of conjugacy classes of  $\mathcal{E}$ .

**Solution:** (**NOT**) We know that  $\mathcal{E}(\{1\}) = \{\{1\}, \{2\}, \{3\}, \{4\}\}$ , of size 4. Since  $|A| = 24$ , there are  $\frac{24}{4} = 6$  conjugacy classes.

**Solution.** The conjugacy classes correspond to \_\_\_\_\_.

## The Equivalence Principle (Group Activity)

**Example.** In how many ways can we arrange 10 people into five pairs?

**Setup:** Let  $A$  be the set of 10-lists  $a = (a_1, a_2, \dots, a_9, a_{10}) \in A$ .

List  $a$  represents the pairings  $\{\{a_1, a_2\}, \dots, \{a_9, a_{10}\}\}$ .

Define lists  $a$  and  $a'$  to be equivalent if the set of pairs is the same.

[*For example, (3, 2, 9, 10, 1, 5, 8, 7, 4, 6)  $\equiv$  (2, 3, 9, 10, 1, 5, 6, 4, 8, 7).*]

(Why is this an equivalence relation?)

**Discuss:** How many different 10-lists are in the same equivalence class?

**Answer:**

By the equivalence principle,