Determinants in the Kronecker product of matrices: The incidence matrix of a complete graph

Christopher R. H. Hanusa[1] and Thomas Zaslavsky[2]

[1] *Department of Mathematics, Queens College (CUNY), 65-30 Kissena Blvd., Flushing, NY 11367, U.S.A. Phone: +1-718-997-5964. Email:* `chanusa@qc.cuny.edu`.
[2] *Department of Mathematical Sciences, Binghamton University (SUNY), Binghamton, NY 13902-6000, U.S.A., Email:* `zaslav@math.binghamton.edu`.

Abstract. We investigate the least common multiple of all subdeterminants, $\mathrm{lcmd}(A \otimes B)$, of a Kronecker product of matrices, of which one is an integral matrix A with two columns and the other is the incidence matrix of a complete graph. We prove that this quantity is the least common multiple of $\mathrm{lcmd}(A)$ to the power $n-1$ and certain binomials in the entries of A.

Résumé. Nous examinons le plus petit commun multiple de tous les sous-déterminants, $\mathrm{lcmd}(A \otimes B)$, d'un produit de Kronecker de matrices. L'une des matrices, A, est à entrées entières à deux colonnes et l'autre est la matrice d'incidence d'un graphe complet. Nous prouvons que le $\mathrm{lcmd}(A \otimes B)$ est égale au plus petit commun multiple du $\mathrm{lcmd}(A)$ à la puissance $n-1$ et de certains binômes des coefficients de A.

Keywords: Kronecker product, determinant, least common multiple, incidence matrix of complete graph, matrix minor

In a study of non-attacking placements of chess pieces, Chaiken, Hanusa, and Zaslavsky [1] were led to a quasipolynomial formula that depends in part on the least common multiple of the determinants of all square submatrices of a certain Kronecker product matrix, namely, the Kronecker product of an integral $m \times 2$ matrix with the incidence matrix of a complete graph. We give a concise expression for the least common multiple of the subdeterminants of this product matrix.

For matrices $A = (a_{ij})_{m\times k}$ and $B = (b_{ij})_{n\times l}$, the Kronecker product $A \otimes B$ is defined to be the $mn \times kl$ block matrix

$$\left[\begin{array}{c:c:c} a_{11}B & \cdots & a_{1k}B \\ \hdashline \vdots & \ddots & \vdots \\ \hdashline a_{m1}B & \cdots & a_{mk}B \end{array}\right].$$

It is known (see [2], for example) that when A and B are square matrices of orders m and n, respectively, then $\det(A \otimes B) = \det(A)^n \det(B)^m$. The quantity we want to compute is $\mathrm{lcmd}(A \otimes B)$, where for an integer matrix M, the notation $\mathrm{lcmd}(M)$ denotes the least common multiple of the determinants of all

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square submatrices. This is a much stronger question, as the matrices A and B are most likely not square and the result depends on all square submatrices of their Kronecker product.

We are interested in the case where A is an integral $m \times 2$ matrix and B is the incidence matrix of a complete graph on n vertices. (Their Kronecker product has order $mn \times 2\binom{n}{2}$.) For a simple graph $G = (V, E)$, the incidence matrix $D(G)$ is a $|V| \times |E|$ matrix with a row corresponding to each vertex in V and a column corresponding to each edge in E. For a column that corresponds to an edge $e = vw$, there are exactly two non-zero entries: one $+1$ and one -1 in the rows corresponding to v and w. The sign assignment is arbitrary.

We concern ourselves with the complete graph K_n, the graph on n vertices $v_1, \ldots, v_n$ with an edge between every pair of vertices. As an example, one of the many incidence matrices for K_4 is the 4×6 matrix

$$D(K_4) = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix};$$

the Kronecker product $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \otimes D(K_4)$ is

$$\left[\begin{array}{cccccc:cccccc} a_{11} & a_{11} & a_{11} & 0 & 0 & 0 & a_{12} & a_{12} & a_{12} & 0 & 0 & 0 \\ -a_{11} & 0 & 0 & a_{11} & a_{11} & 0 & -a_{12} & 0 & 0 & a_{12} & a_{12} & 0 \\ 0 & -a_{11} & 0 & -a_{11} & 0 & a_{11} & 0 & -a_{12} & 0 & -a_{12} & 0 & a_{12} \\ 0 & 0 & -a_{11} & 0 & -a_{11} & -a_{11} & 0 & 0 & -a_{12} & 0 & -a_{12} & -a_{12} \\ \hdashline a_{21} & a_{21} & a_{21} & 0 & 0 & 0 & a_{22} & a_{22} & a_{22} & 0 & 0 & 0 \\ -a_{21} & 0 & 0 & a_{21} & a_{21} & 0 & -a_{22} & 0 & 0 & a_{22} & a_{22} & 0 \\ 0 & -a_{21} & 0 & -a_{21} & 0 & a_{21} & 0 & -a_{22} & 0 & -a_{22} & 0 & a_{22} \\ 0 & 0 & -a_{21} & 0 & -a_{21} & -a_{21} & 0 & 0 & -a_{22} & 0 & -a_{22} & -a_{22} \end{array}\right].$$

We now introduce our main result after some notation.

Let $A = (a_{ij})$ be an $m \times 2$ matrix; this makes $A \otimes D(K_n)$ an $mn \times n(n-1)$ matrix with non-zero entries $\pm a_{ij}$. We introduce new notation for some matrices that will arise naturally in our theorem. For $i, j \in [m] := \{1, 2, \ldots, m\}$, we write $A^{i,j}$ to represent $\begin{pmatrix} a_{i1} & a_{i2} \\ a_{j1} & a_{j2} \end{pmatrix}$. If I is a multisubset of $[m]$, we define a_{Ik} to be the product $\prod_{i \in I} a_{ik}$. If I and J are multisubsets of $[m]$, we define $A^{I,J}$ to be the matrix $\begin{pmatrix} a_{I1} & a_{I2} \\ a_{J1} & a_{J2} \end{pmatrix}$. In this notation,

$$\operatorname{lcmd} A = \operatorname{lcm}\big(\operatorname*{LCM}_{i,k} a_{ik}, \operatorname*{LCM}_{i,j} \det A^{i,j}\big),$$

where LCM denotes the least common multiple of non-zero quantities taken over all indicated pairs of indices.

Theorem 1 *Let A be an $m \times 2$ matrix, not identically zero, and $n \geq 1$. The least common multiple of all square minor determinants of $A \otimes D(K_n)$ is*

$$\operatorname{lcmd}\big(A \otimes D(K_n)\big) = \operatorname{lcm}\left((\operatorname{lcmd} A)^{n-1}, \operatorname*{LCM}_{\mathcal{K}}\left(\prod_{(I_s,J_s)\in\mathcal{K}} \det A^{I_s,J_s}\right)\right), \tag{1}$$

where LCM *denotes the least common multiple of non-zero quantities taken over all collections* $\mathcal{K} = \{(I_s, J_s)\})$ *of pairs of multisubsets* I_s *and* J_s *of* $[m]$ *satisfying* $|I_s| = |J_s|$ *and* $I_s \cap J_s = \emptyset$ *for all* s *and* $2\sum |I_s| \leq n$.

It is only necessary to take the LCM component over all maximal collections $\mathcal{K}$, that is, collections $\mathcal{K}$ satisfying $\sum |I_s| = \lfloor n/2 \rfloor$.

Proof: Consider an $l \times l$ submatrix N of $A \otimes D(K_n)$. We wish to determine the determinant of N and show that it divides the right-hand side of Equation (1). We need consider only matrices N whose determinant is not zero, since a matrix with $\det N = 0$ has no effect on the least common multiple.

Since $D(K_n)$ is constructed from a graph, we will analyze N from a graphic perspective. The matrix N is a choice of l rows and l columns from $A \otimes D(K_n)$. This corresponds to a choice of l vertices and l edges from K_n where we are allowed to choose up to m copies of each vertex and up to two copies of an edge. Another way to say this is that we are choosing m subsets of $V(K_n)$, say V_1 through V_m, and two subsets of $E(K_n)$, say E_1 and E_2, with the property that $\sum_{i=1}^{m} |V_i| = \sum_{k=1}^{2} |V_k| = l$. From this point of view, if a row in N is taken from the first n rows of $A \otimes D(K_n)$, this is thought of as placing the corresponding vertex of $V(K_n)$ in V_1, and so on, up through a row in N from the last n rows of $A \otimes D(K_n)$ corresponding to a vertex in V_m. We will say that the copy of v in V_i is the ith *copy of* v and the copy of e in E_k is the kth *copy of* e.

Under this framework, we will now perform elementary matrix operations on N in order to make its determinant easier to calculate. We call the resulting matrix the *simplified matrix of* N. Each copy of a vertex v has a row in N associated with it; two rows corresponding to two copies of the same vertex contain the same entries except for the different multipliers a_{ik}. For example, if v is a vertex in both V_1 and V_2, then there is a row corresponding to the first copy with multipliers a_{11} and a_{12} and a row corresponding to the second copy with the same entries multiplied by a_{21} and a_{22}.

In the case when there is a vertex in exactly two vertex sets V_i and V_j corresponding to two rows R_i and R_j in N, we perform the following operations depending on the multipliers a_{i1}, a_{i2}, a_{j1}, and a_{j2}. We first notice that $\det A^{i,j} = a_{i1}a_{j2} - a_{i2}a_{j1}$ is non-zero; otherwise, the rows R_i and R_j would be linearly dependent in N and $\det N = 0$. Therefore either both a_{i1} and a_{j2} or a_{i2} and a_{j1} are non-zero. In the former case, let us add $-a_{j1}/a_{i1}$ times R_i to R_j in order to zero out the entries corresponding to edges in E_1. The multipliers of entries in R_j corresponding to edges in E_2 are now all $\det A^{i,j}/a_{i1}$. Similarly, we can zero out the entries in R_i corresponding to edges in E_2. Lastly, factor out $\det A^{i,j}/a_{j2}a_{i1}$ from R_j. If on the other hand, either multiplier a_{i1} or a_{j2} is zero, then reverse the roles or i and j in the preceding argument. There cannot be a vertex in three or more vertex sets since then the corresponding rows of N would be linearly dependent and $\det N$ would be zero. The above manipulations ensure that the multiplier of every non-zero entry in N that corresponds to an ith vertex and a kth edge is a_{ik}.

We assert that the simplified matrix of N has no more that two non-zero entries in any column. For a column e corresponding to an edge $e = vw$ in K_n, each of v and w is either in one vertex set V_i or in two vertex sets V_i and V_j. If the vertex corresponds to two rows in N, the above manipulations ensure that there is only one copy of the vertex that has a non-zero multiplier in the column. Another important quality of this simplification is that if a vertex is in more than one vertex set, then every edge incident with once instance of this repeated vertex is now in the same edge set.

Since we are assuming $\det N \neq 0$, N has at least one non-zero entry in each column or row. If a row (or column) has exactly one non-zero entry, we can reduce the determinant by expanding in that row (or column). This contributes that non-zero entry as a factor in the determinant. After reducing repeatedly

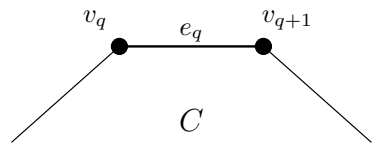

Fig. 1: An edge $e_q = v_q v_{q+1}$ in the cycle C generated by block B. When $v_q \in V_i$, $v_{q+1} \in V_j$, and $e_q \in E_k$, the contributions y_q and z_q to $\det B$ are a_{ik} and a_{jk}, respectively.

in this way, we arrive at a matrix where each column has exactly two non-zero entries, and each row has at least two non-zero entries. This implies that every row has exactly two non-zero entries as well. After interchanging the necessary columns and rows and possibly multiplying columns by -1, the structure of what we will call the *reduced matrix of* N is a block diagonal matrix where each block B is a weighted incidence matrix of a cycle, such as

$$\begin{bmatrix} y_1 & 0 & 0 & 0 & 0 & -z_6 \\ -z_1 & y_2 & 0 & 0 & 0 & 0 \\ 0 & -z_2 & y_3 & 0 & 0 & 0 \\ 0 & 0 & -z_3 & y_4 & 0 & 0 \\ 0 & 0 & 0 & -z_4 & y_5 & 0 \\ 0 & 0 & 0 & 0 & -z_5 & y_6 \end{bmatrix}.$$

The determinant of a $p \times p$ matrix of this type is $y_1 \cdots y_p - z_1 \cdots z_p$. Therefore, we can write the determinant of N as the product of powers of entries of A, powers of $\det A^{i,j}$, and binomials of this form.

In our situation, the entries y_q and z_q are the variables a_{ik}, depending on in which vertex sets the rows lie and in which edge sets the columns lie. If the vertices of K_n corresponding to the rows in B are labeled v_1 through v_p, this block of the block matrix corresponds to traversing the closed walk $C = v_1 v_2 \ldots v_p v_1$ in K_n (in this direction). As a result of the form of the simplified matrix of N, for a column that corresponds to an edge $e_q = v_q v_{q+1}$ in E_k traversed from the vertex v_q in vertex set V_i to the vertex v_{q+1} in vertex set V_j, the entry y_q is a_{ik} and the entry z_q is a_{jk}. (See Figure 1.) Therefore each block B in the block diagonal matrix contributes

$$\det B = \prod_{\substack{e=v_q v_{q+1} \in C \\ e \in E_k, v_q \in V_i}} a_{ik} - \prod_{\substack{e=v_q v_{q+1} \in C \\ e \in E_k, v_{q+1} \in V_j}} a_{jk} \tag{2}$$

for some closed walk C in G.

We can simplify this expression by analyzing what exactly the a_{ik} and a_{jk} are. Suppose that two adjacent edges e_{q-1} and e_q in C are in the same edge set E_k, and suppose that the vertex v_q that these edges share is in V_i. (See Figure 2.) In this case, both entries z_{q-1} and y_q are a_{ik}, which can then be factored out of each product in Equation (2). A particular case to mention is when the cycle C contains a vertex that has multiple copies in N (not necessarily both in C). In this case, the edges of C incident with this repeated vertex are both from the same edge set, as mentioned earlier. After factoring out a multiplier for each pair of adjacent edges in the same edge set, all that remains inside the products in Equation (2) are the contributions of multipliers from vertices where the incident edges are from different edge sets.

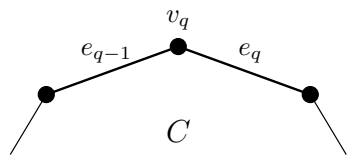

Fig. 2: Two adjacent edges e_{q-1} and e_q, both incident with vertex v_q in the cycle C generated by block B. When both edges are members of the same edge set E_k and v_q is a member of V_i, the contributions z_{q-1} and y_q are both a_{ik}, allowing this multiplier to be factored out of Equation (2).

More precisely, when following the closed walk, let I be the multiset of indices i such that the walk C passes from an edge in E_2 to an edge in E_1 at a vertex in V_i. Similarly, let J be the multiset of indices j such that C passes from an edge in E_1 to an edge in E_2 at a vertex in V_j. Then what remains inside the products in Equation (2) after factoring out common multipliers is exactly

$$\det A^{I,J} = \prod_{i\in I} a_{i1} \prod_{j\in J} a_{j2} - \prod_{j\in J} a_{j2} \prod_{i\in I} a_{i2}.$$

There is one final simplifying step. Consider a value i occurring in both I and J. In this case, we can factor $a_{i1}a_{i2}$ out of both terms. This implies that the determinant of each block B of the block diagonal matrix is of the form

$$\pm\left(\prod_{i,k} a_{ik}^{s_{ik}}\right) \det A^{I,J}, \tag{3}$$

where the exponents $s_{i,k}$ are non-negative integers and I and J are disjoint subsets of $[m]$ of the same cardinality and $2|I| + \sum_{i,k} s_{ik} = p$ because the degree of $\det B$ is the order of B. Notice that when $|I| = |J| = 1$ (say $I = \{i\}$ and $J = \{j\}$), the factor $\det A^{I,J}$ equals $\det A^{i,j}$. Combining contributions from the simplification process and all blocks, we have

$$\det N = \pm \prod_{i,j} (\det A^{i,j})^{|V_i \cap V_j|} \prod_{i,k} a_{ik}^{S_{ik}} \prod_{B} \det A^{I_B, J_B},$$

for some non-negative exponents S_{ik}.

We now verify that this product divides the right hand side of Equation (1). The exponents S_{ik} can be no larger than n because there are only n rows with entries a_{ik} in $A \otimes D(K_n)$, so as a polynomial in the variable a_{ik}, every term in the expansion of the determinant has degree at most n. Furthermore, it is not possible for the exponent of any a_{ik} to be n. The only way this might occur is if N were to contain all n vertices of V_i and at least n edges of E_k incident with the vertices of V_i. The corresponding set of columns is a dependent set of columns in N (the rank of $D(K_n)$ is n), which would make $\det N = 0$. Therefore $\det N$ contributes no more than $n-1$ factors of any a_{ik} to $\operatorname{lcmd}(A \otimes D(K_n))$.

Now let us determine the number of factors of $\det A^{i,j}$ that may divide $\det N$. Notice that factors of $\det A^{i,j}$ do not solely arise upon the conversion of N to the simplified matrix of N—they also arise when $|I| = |J| = 1$ in Equation (3). We need to take this into account to determine how many factors of $\det A^{i,j}$ may divide $\det N$. It is not possible for n factors $\det A^{i,j}$ to arise through the initial simplification of N.

If N contained all rows corresponding to vertices of V_i and V_j along with $2n$ columns of $A \otimes D(K_n)$, then at least one edge set (E_1 or E_2) would contain n edges from K_n. The columns corresponding to these edges would form a dependent set of columns in N, making $\det N = 0$. Because every $\det A^{i,j}$ contribution from blocks must include at least one vertex in V_i with no copies and one vertex in V_j with no copies, there can be no more than $n - 1 - |V_i \cap V_j|$ copies of $\det A^{i,j}$ from blocks. Therefore there cannot be more than $n - 1$ factors $\det A^{i,j}$ in the contribution of $\det N$ to $\operatorname{lcmd}(A \otimes D(K_n))$.

The factors of $\det N$ from the blocks B of the reduced matrix of N can be considered independently from the contributions of a_{ik} and $\det A^{i,j}$ to Equation (1). There may be contributions of $\det A^{i,j}$ in both terms of the right hand side of Equation (1); however, they will not occur twice after application of the outer lcm operation. Regarding the contribution of blocks B, the number of possible $\det A^{I,J}$ factors is limited as well. For every factor $\det A^{I,J}$, each vertex leading to an index in I and every vertex leading to an index in J must have no additional copies in the graph. Hence each vertex of K_n occurs at most once in some closed walk C generated by a block B. Depending on the vertex set into which this vertex is placed, the vertex either contributes an index to either I_B or J_B. So we are in a situation where for all B, $|I_B| = |J_B|$, $I_B \cap J_B = \emptyset$, and $\sum_B (|I_B| + |J_B|) \leq n$, the product of which is the exact description of one product in the $\mathrm{LCM}_{\mathcal{K}}$ contribution of Equation (1)

We have shown that for every matrix N, $\det N$ divides the right-hand side of Equation (1). We now show that there exist graphs that attain the claimed powers of factors. Consider the path of length $n-1$, $P = v_1 v_2 \cdots v_n$, as a subgraph of K_n. Create the $(2n-2) \times (2n-2)$ submatrix N of $A \otimes D(K_n)$ with rows corresponding to both an ith copy and a jth copy of vertices v_1 through v_{n-1} and edges corresponding to two copies of every edge in P. Then

$$N = \left[\begin{array}{cccc:cccc}
a_{i1} & 0 & 0 & 0 & a_{i2} & 0 & 0 & 0 \\
-a_{i1} & a_{i1} & 0 & 0 & -a_{i2} & a_{i2} & 0 & 0 \\
0 & \ddots & \ddots & 0 & 0 & \ddots & \ddots & 0 \\
0 & 0 & -a_{i1} & a_{i1} & 0 & 0 & -a_{i2} & a_{i2} \\
\hdashline
a_{j1} & 0 & 0 & 0 & a_{j2} & 0 & 0 & 0 \\
-a_{j1} & a_{j1} & 0 & 0 & -a_{j2} & a_{j2} & 0 & 0 \\
0 & \ddots & \ddots & 0 & 0 & \ddots & \ddots & 0 \\
0 & 0 & -a_{j1} & a_{j1} & 0 & 0 & -a_{j2} & a_{j2}
\end{array}\right],$$

with determinant $(\det A^{i,j})^{n-1}$. The four quadrants of N are $(n-1) \times (n-1)$ submatrices of $A \otimes D(K_n)$ with determinants a_{i1}^{n-1}, a_{i2}^{n-1}, a_{j1}^{n-1}, and a_{j2}^{n-1}, respectively.

For all collections $\mathcal{K} = \{(I_s, J_s)\})$ of pairs of multisubsets I_s and J_s of $[m]$ satisfying $|I_s| = |J_s|$ and $I_s \cap J_s = \emptyset$ for all s and $2\sum |I_s| \leq n$, we show that there is a submatrix N of $A \otimes D(K_n)$ with determinant $\prod_{(I_s,J_s)\in\mathcal{K}} \det A^{I_s,J_s}$. For all s starting with $s = 1$, choose a subgraph of K_n with $2|I_s|$ vertices as follows. If $|I_s| > 1$, create a cycle C_s of length $2|I_s|$ using the next $|I_s|$ unused vertices of K_n. If $|I_s| = 1$ and the first unused vertex is vertex v_{q+1}, then just create the edge $e_s = v_{q+1}v_{q+2}$. Choose the rows and columns of $A \otimes D(K_n)$ for N using the framework of placing the vertices and edges of K_n into vertex sets V_i and edge sets E_k. For the cycle C_s with vertices $v_{q+1}, v_{q+2}, \ldots, v_{q+2|I_s|}$, place the odd-indexed vertices into a vertex set V_i for every $i \in I_s$ and the even-indexed vertices into a vertex set V_j for $j \in J_s$. Place the edges from a lower-indexed odd vertex to a higher-indexed even vertex in E_1 and

place all other edges in E_2. When $I_s = \{i\}$ and $J_s = \{j\}$, place vertex v_{q+1} in V_i and vertex v_{q+2} in V_j. Place the edge $v_{q+1}v_{q+2}$ in both E_1 and E_2.

The submatrix N of $A \otimes D(K_n)$ that arises from placing the vertices in lexicographic order and the edges in order following the cycles C_s or edge e_s is the block-diagonal matrix with blocks N_s where each matrix N_s is a $2|I_s| \times 2|I_s|$ matrix of the form

$$\begin{bmatrix} a_{i_1 1} & 0 & 0 & \cdots & 0 & a_{i_1 2} \\ -a_{j_1 1} & -a_{j_1 2} & 0 & \cdots & 0 & 0 \\ 0 & a_{i_2 2} & a_{i_2 1} & & & 0 \\ 0 & 0 & -a_{j_2 1} & -a_{j_2 2} & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & & -a_{j_l 1} & -a_{j_l 2} \end{bmatrix}$$

if C_s is a cycle and

$$\begin{pmatrix} a_{i1} & a_{i2} \\ a_{j1} & a_{j2} \end{pmatrix}$$

if e_s is an edge. The determinant of N_s is exactly $\det A^{I_s,J_s}$ for all s, so the determinant of N is $\prod_{(I_s,J_s)\in\mathcal{K}} \det A^{I_s,J_s}$, as desired. □

When understanding the right hand side of Equation (1), it may be instructive to notice that the LCM factor on the right hand side divides

$$\prod_{\substack{\text{disjoint } I,\, J: \\ |I|=|J|=p}} (\det A^{I,J})^{\lfloor n/2p \rfloor},$$

as the largest number of individual $\det A^{I,J}$ factors that may occur for disjoint p-member multisubsets I and J of $[m]$ is $\lfloor n/2p \rfloor$.

When $m = 2$, the only pair of disjoint p-member multisubsets of $[m]$ is $\{1^p\}$ and $\{2^p\}$. From this, we have the following corollary.

Corollary 2 *Let A be a 2×2 matrix, not identically zero, and $n \geq 1$. The least common multiple of all square minor determinants of $A \otimes D(K_n)$ is*

$$\operatorname{lcmd}\big(A \otimes D(K_n)\big) = \operatorname{lcm}\Big((\operatorname{lcmd} A)^{n-1}, \operatorname*{LCM}_{p=2}^{\lfloor n/2 \rfloor} \big((a_{11}a_{22})^p - (a_{12}a_{21})^p\big)^{\lfloor n/2p \rfloor}\Big),$$

where LCM *denotes the least common multiple over the range of* p.

We calculate a few examples, with matrices A that are needed for the chess-piece problem of [1].

Example 1 When the chess piece is the bishop, A is the 2×2 matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

We apply Corollary 2, noting that $\operatorname{lcmd}(A) = 2$. We get

$$\operatorname{lcmd}\big(A \otimes D(K_n)\big) = \operatorname{lcm}\big(2^{n-1}, \operatorname*{LCM}_{p=2}^{\lfloor n/2 \rfloor} \big((-1)^p - 1^p\big)^{\lfloor n/2p \rfloor}\big).$$

The LCM generates powers of 2 no larger than $2^{n/2}$, hence $\operatorname{lcmd}\big(A \otimes D(K_n)\big) = 2^{n-1}$.

Example 2 When the chess piece is the queen, A is the 4×2 matrix

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Again, $\operatorname{lcmd}(A) = 2$. We apply Theorem 1. Every pair (I, J) of disjoint p-member multisubsets of $[4]$ has one of the following seven forms, up to the order of I and J:

$$(\{1^q\}, \{2^r, 3^s, 4^t\}), \quad (\{2^r\}, \{1^q, 3^s, 4^t\}), \quad (\{3^s\}, \{1^q, 2^r, 4^t\}), \quad (\{4^t\}, \{1^q, 2^r, 3^s\}),$$
$$(\{1^q, 2^r\}, \{3^s, 4^t\}), \quad (\{1^q, 3^s\}, \{2^r, 4^t\}), \quad (\{1^q, 4^t\}, \{2^r, 3^s\}),$$

where the sum of the exponents in each multisubset is p, and where q, r, s, and t may be zero. When we calculate $\det A^{I,J}$ for each pair (I, J), the presence of zeroes and ones in A simplifies our calculations. For example, in the third case, where $I = \{3^s\}$ and $J = \{1^q, 2^r, 4^t\}$, the matrix $A^{I,J}$ is as follows:

$$A^{I,J} = \begin{pmatrix} 1^s & 1^s \\ 1^q 0^r 1^t & 0^q 1^r (-1)^t \end{pmatrix},$$

where $s = q + r + t = p$. If any entry in this matrix is zero, then the determinant is a product of 0's, 1's, and -1's, which will not contribute to the LCM. Therefore the only non-trivial case is when $q = r = 0$, in which case $s = t = p$, and $\det A^{I,J} = (-1)^p - 1^p = 0$ or -2. This implies that the LCM of Equation (1) divides 2^{n-1}. We conclude that $\operatorname{lcmd}\big(A \otimes D(K_n)\big) = 2^{n-1}$.

Example 3 A more difficult example is the fairy chess piece known as a nightrider, which moves an unlimited distance in the directions of a knight. Here A is the 4×2 matrix

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 1 & -2 \\ 2 & -1 \end{pmatrix}.$$

The submatrices

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}, \text{ and } \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix},$$

with determinants -3, -4, and -5, respectively, lead to the conclusion that $\operatorname{lcmd}(A) = 60$. Since the dimensions of A are as in Example 2, we have the same possibilities for disjoint pairs of p-member multisubsets of $[4]$. It turns out that $\det A^{I,J}$ has the same form in all seven cases: precisely $\pm 2^u(2^{2p-2u} \pm 1)$, where u is a number between 0 and p. Furthermore, every value of u from 0 to p appears and every

choice of plus or minus sign appears (except when $u = p$) in $\det A^{I,J}$ for some choice of (I, J). We present two representative examples that support this assertion.

The case of $I = \{1^q\}$ and $J = \{2^r, 3^s, 4^t\}$. Then

$$A^{I,J} = \begin{pmatrix} 1^q & 2^q \\ 2^r 1^s 2^t & 1^r(-2)^s(-1)^t \end{pmatrix},$$

with $q = r + s + t = p$. We can rewrite $\det A^{I,J}$ as $\pm 2^s - 2^{2p-s} = -2^s(2^{2p-2s} \pm 1)$. The only instance in where there is no choice of sign is when $s = p$ and $r = t = 0$, in which case $\det A^{I,J}$ simplifies to either 0 or -2^{p+1}.

The case of $I = \{1^q, 2^r\}$ and $J = \{3^s, 4^t\}$. Then

$$A^{I,J} = \begin{pmatrix} 1^q 2^r & 2^q 1^r \\ 1^s 2^t & (-2)^s(-1)^t \end{pmatrix},$$

where $q + r = s + t = p$. For this choice of I and J, $\det A^{I,J} = (-1)^p 2^{r+s} - 2^{2p-r-s}$.

Since every $\det A^{I,J}$ has the same form, and at most $\lfloor p/2n \rfloor$ factors of type $(2^{2p-2u} \pm 1)$ may occur at the same time, the LCM in Equation (1) is exactly

$$\operatorname*{LCM}_{\mathcal{K}} \left(\prod_{(I_s,J_s)\in\mathcal{K}} \det A^{I_s,J_s} \right) = 2^N \operatorname*{LCM}_{\substack{1\le p\le n/2 \\ 0\le u\le p-1}} (2^{2p-2u} \pm 1)^{\lfloor n/2p \rfloor},$$

for some $N \le n$. We conclude that

$$\operatorname{lcmd}\big(A \otimes D(K_n)\big) = \operatorname{lcm}(60^{n-1}, \operatorname*{LCM}_{\substack{1\le p\le n/2 \\ 0\le u\le p-1}} (2^{2p-2u} \pm 1)^{\lfloor n/2p \rfloor}).$$

As a sample of the type of answer we get, when $n = 8$ this expression is

$$\begin{aligned} \operatorname{lcmd}\big(A \otimes D(K_8)\big) &= \operatorname{lcm}(60^7, (4 \pm 1)^{\lfloor 8/2 \rfloor}, (16 \pm 1)^{\lfloor 8/4 \rfloor}, (64 \pm 1)^{\lfloor 8/6 \rfloor}, (256 \pm 1)^{\lfloor 8/8 \rfloor}) \\ &= 60^7 \cdot 7 \cdot 13 \cdot 17^2 \cdot 257. \end{aligned}$$

The first few values of n give the following numbers:

n	$\operatorname{lcmd}\big(A \otimes D(K_8)\big)$	(factored)
2	60	60^1
3	3600	60^2
4	3672000	$60^3 \cdot 17$
5	220320000	$60^4 \cdot 17$
6	1202947200000	$60^5 \cdot 7 \cdot 13 \cdot 17$
7	72176832000000	$60^6 \cdot 7 \cdot 13 \cdot 17$
8	18920434740480000000	$60^7 \cdot 7 \cdot 13 \cdot 17^2 \cdot 257$
9	1135226084428800000000	$60^8 \cdot 7 \cdot 13 \cdot 17^2 \cdot 257$
10	952295753183943168000000000	$60^9 \cdot 7 \cdot 11 \cdot 13 \cdot 17^2 \cdot 31 \cdot 41 \cdot 257$

We hope to determine in the future whether $\operatorname{lcmd}(A \otimes B)$ has a simple form for arbitrary matrices A and B. Our limited experimental data suggests this may be difficult. However, we think at least some generalization of Theorem 1 is possible.

Another direction worth investigating is the number theoretic aspects of Theorem 1.

Our initial goal was a formula for $\mathrm{lcmd}(A \otimes D(K_n))$. Theorem 1 gives a compact expression for this quantity, but not as simple as it could be without the least common multiples. Improving it would require an understanding of when two multivariate binomials have a common divisor.

References

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