An Exploration of Aztec Pillows

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TABLE OF CONTENTS

LIST OF FIGURES

Chapter 1

INTRODUCTION

The subject area of my work is the study of enumeration of tilings of regions, or equivalently matchings of certain bipartite graphs. In Chapter 1 of this paper, I present a brief history of the subject. In Chapter 2, I specify the general aim of my research and define the idea of Aztec Pillows. In Chapter 3, I present some preliminary results. Lastly, in Chapter 4, I expose various open questions in the vein of the research. Research questions that arise will be presented throughout the paper as they become posable.

1.1 Tilings and Matchings

Throughout this paper, we will be enumerating the number of complete tilings of regions with dominoes. In general, consider any tiling of the plane with polygons. We consider a diform to be the union of two adjacent polygons. In the particular case when the polygons are all squares, these diforms are called dominoes. As an example, consider a chessboard. A domino will be the union of two adjacent squares.

A tiling of the region will consist of an arrangement of non-overlapping diforms which cover all polygons on the board. In our example, this implies that we use 32 non-overlapping dominoes to cover the chessboard.

Another way to think of a complete tiling of the region is to consider a perfect matching of the dual graph of the region (excluding the outer face). In our chessboard example, we have 64 vertices which represent the squares of the chessboard, and 112 edges representing the adjacencies of the squares. We wish to count the number of ways that 32 of these edges form a perfect matching. In terms of this research, we shall never create a partial matching, so after now any reference to a "matching" is a reference to a "perfect matching".

In this paper, to abbreviate "the number of tilings of" in formulas, I will use # notation, as in $\#AZ_n$. This notation appears in [22] by Pachter.

1.2 History

The problem in its current form originated in physics and chemistry in the 1930's [25]. Physicists were looking for a model of the liquid and gaseous states, by considering diatomic molecules (dimers) as edges in the square lattice. For this reason, the model is sometimes called the dimer model. Chemists were interested in aromatic hydrocarbons; hydrocarbons form a honeycomb grid, and double bonds need to be placed in this lattice such that each vertex has exactly one double bond attached to it. (See Figure 1.1 for examples of these graphs.)

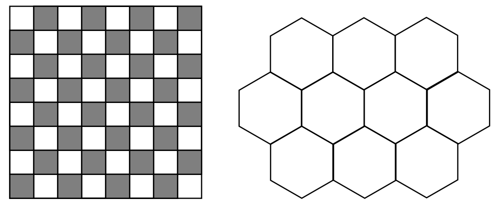

Figure 1.1: Rectangular Board and Honeycomb Graph

Progress was made with the infinite square grid mid-century, as Kasteleyn was able to count the number of tilings of a rectangular region as well as the associated torus. Given an $n \times m$ board, with both n and m even, there are

$$\prod_{j=1}^{n/2}\prod_{k=1}^{m/2}\left(4\cos^2\frac{\pi j}{n+1}+4\cos^2\frac{\pi k}{m+1}\right)$$

domino tilings of the board [14]. This result involves products of cosines and is not obviously an integer. If we consider only the one-parameter family of graphs that is an $2n \times 2n$ subgraph S_n of the infinite square lattice, we see that its formula has even more structure, being either a perfect square or two times a perfect square.

Using Galois Theory, you can prove that it is an integer [4], but this is less than appealing. The region seems "nice", so there should be a simple proof of this result. In 1996 Pachter [22], based on work by Ciucu [5], found a nice way to decompose the square region S_n into two congruent regions H_n and prove combinatorially that $\#S_n = 2^n(\#H_n)^2$. This result is much more satisfactory, because it explains the formula much more clearly than just "a perfect square or two times a perfect square". As for the general rectangle case, the formula becomes no simpler, but thanks to Percus [23] can be proved using determinants by exploiting the bipartite nature of the graph instead of with the Pfaffian method that Kasteleyn used initially.

As for the honeycomb grid, it has been studied in depth as well. For a combinatorialist, the most interesting aspect about the matchings on a honeycomb grid or lozenge tilings of a hexagon is their many combinatorial forms. They relate to plane partitions and thus to solid Young diagrams [6, 9]. Lozenge tilings of an equiangular hexagon with side lengths (a, b, c) are in one to one correspondence with a solid Young diagram on $[0, a] \times [0, b] \times [0, c]$. This correspondence can be understood visually by thinking of unit cubes fitting inside a box of size $a \times b \times c$, and looking at the box from a point far away, gives the appearance of a hexagon with rhombi tiling it. For an example of this visualization, see Figure 1.2.

At MIT in the 1990's, Jim Propp organized the Undergraduate Research Project in Random Tilings. (Archived website: http://www.math.wisc.edu/˜propp/tiling/www/). The goal of this program was to understand more fully random tilings of Aztec diamonds (see next section) and other regions. This program had over 50 participants in its five years, including Henry Cohn. After moving to the University of Wisconsin, Jim Propp wrote a

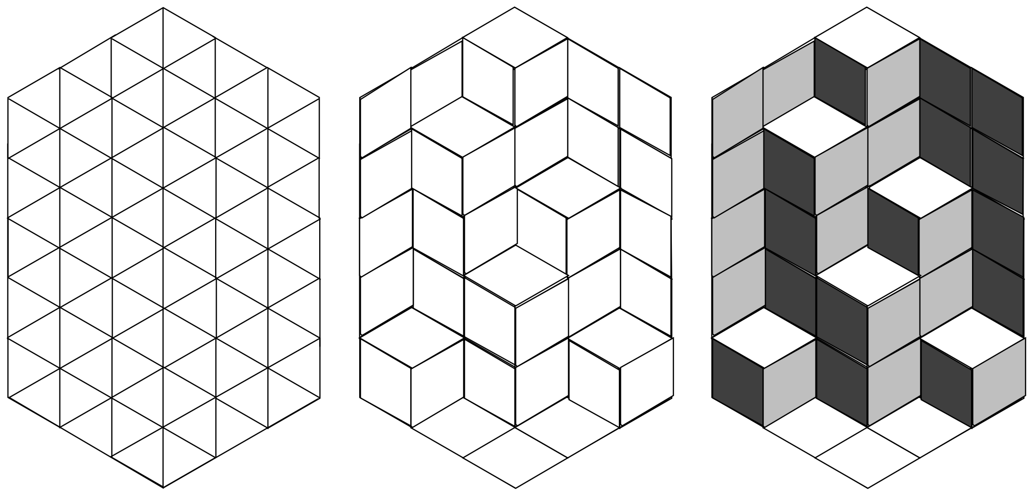

Figure 1.2: A Lozenge Tiling on a Triangular Grid and the Associated Plane Partition

survey paper on diform tilings in 1999. This survey included a summary of research in this field and thirty-two problems to answer [25]. This is the paper to read if you want more information about diform tilings and the lines of research considered. In addition, Jim Propp maintains a mailing list focusing on domino tilings. With all this organization, the domino tiling community is well established.

1.3 Current Lines of Approach

When rectangular regions were first considered, they were chosen for study because they seemed to be the most natural regions — the regions that would minimize edge effects on tilings. Another region proved to be much simpler to explain, however. In the 1980's, physicists Grensing, Carlsen, and Zapp presented a regularly bounded region that had a nice formula for its number of tilings, but gave no proof of the formula [11]. Later, mathematicians Elkies, Kuperberg, Larsen, and Propp [8] rediscovered this region and gave four proofs of the formula

$$\#AZ_n = 2^{n(n+1)/2}. \tag{1.1}$$

This region was called the Aztec diamond (see Figure 1.3 for an example), denoted by AZ_n, where n is the number of steps. We define a "step" to be a movement upward along one square, followed by right movement until the next step begins. (This clarification will be more important in later sections.)

One reason why this region is believed to have such a nice formula is because of the restrictions the edge effects have on the type of tilings that appear in this region. In particular, it has to do with a height function along the boundary. After coloring the Aztec diamond like a chessboard so that the left-most square in the upper row is black, you can consider the height of a tiling to be increasing incrementally if you follow edges of a black

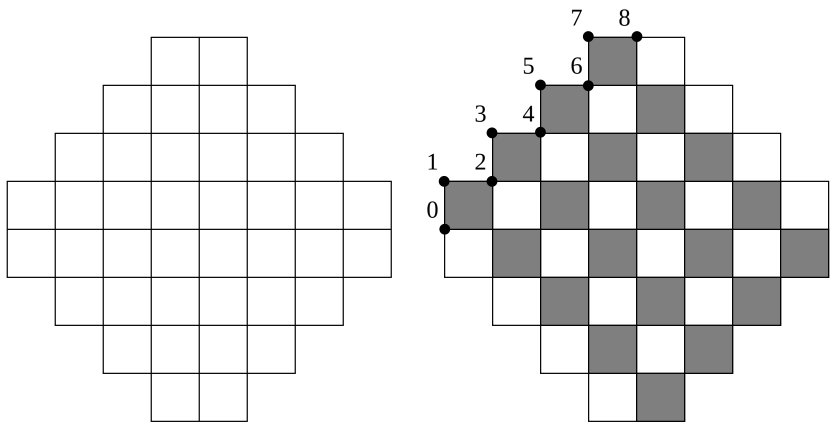

Figure 1.3: An Aztec Diamond (AZ_4) and its Associated Height Function

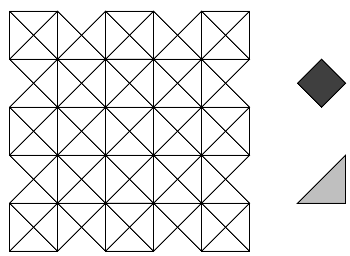

Figure 1.4: A Fortress and Its Diforms

square clockwise or those of a white square counterclockwise (and decreasing if you change directions). In an Aztec diamond, these values steadily increase as you go up the top left diagonal (see Figure 1.3), so that between the top to the bottom of the Aztec diamond there is a very large difference in height, putting extra restrictions on the tiles that can be placed in the region.

For a nice visualization of how height functions restrain tilings, see the Monthly article by Thurston [31]. Height functions also arise in the honeycomb grid; they basically come from the height of blocks in the plane partition corresponding to the tiling. For more information on height functions, see [6, 15, 27].

Diform tiling regions whose diforms are not always congruent are also studied. One example of such a region is called a fortress (see Figure 1.4).

One common aspect to all these regions is that the graphs are all bipartite. Limited research has been done in the non-bipartite case, but it is also based on matrix methods introduced by Kasteleyn.

1.4 Proof Techniques

Various useful techniques have been created by the domino community, and we are always on the lookout for more.

In counting matchings on graphs directly, sometimes it is useful to reduce one graph to another whose number of tilings is known. A tool in this vein is called Urban Renewal, presented by Jim Propp [26]. It uses the idea that you can replace one set of eight weighted edges by another set of four differently weighted edges, and when you count the weighted matchings of the first graph, it equals the number of weighted matchings of the second graph. This is a nice combinatorial technique that sometimes proves useful.

Kasteleyn uses 0-± 1-$\pm i$ matrices in his work, a technique that is the basis for much work in the subject. Helfgott exploits this in his bachelor thesis work, and his result is presented in Section 3.2. Kuperberg uses representation theory in his 1998 and 2001 articles [18, 19]. Jockusch uses combinatorial methods that exploit 2-fold and 4-fold rotational symmetry in his 1994 article [13]. Ciucu uses matching generating functions in graphs with reflective symmetry in his 1996 article [5]. Kuo uses graphical condensation in his 2003 article [17].

A number of tools that have been created to take advantage of the bipartite nature of the graph. One that I used many times is called vaxmacs, a customized emacs environment written by David Wilson, into which you input a graph. This graph is written in VAX format, so called because it is full of V's, A's, and X's. It passes the corresponding Kasteleyn matrix to Maple, which takes the determinant, giving the number of tilings of the region. All the above types of graphs are representable in VAX-code, so creating a sequence of graphs to feed into vaxmacs makes data collection easier. The software and documentation for vaxmacs can be found at http://www.math.wisc.edu/˜propp/software.html.

Once one collects the data, it is nice to be able to find a pattern in the data, and a couple of options are available to that end. The most famous is the On-Line Encyclopedia of Integer Sequences ([28]), but a second option exists in a Mathematica program called RATE (German for guess), where you can input the first terms of a sequence, and it will guess the next terms. The code can be found at http://www.mat.univie.ac.at/˜kratt/rate/rate.html. Combinatorial arguments from Concrete Mathematics [10] and Proofs That Really Count [3] have been especially useful in the proofs I provide.

Chapter 2

PROBLEM STATEMENT

In this chapter, we define the object whose domino tilings we enumerate, and explain some tools that we will use in future chapters.

2.1 Aztec Pillows

An Aztec pillow, as it was initially presented in [25], is a rotationally symmetric region in the plane that has side constraints like those of Aztec diamonds. On the top left diagonal however, the steps are comprised of three squares to the right for every square up. As an example, Figure 2.1 presents a 2 mod 4 pillow of order 4 and a 0 mod 4 pillow of order 5. The "mod" has to do with the number of tiles there are in the top row of the pillow, and the "order" has to do with how many steps are taken.

Aztec pillows were singled out as interesting regions because the number of tilings is conjectured to be a larger number squared times a smaller number with a simple generating function. For 0 mod 4 pillows, the generating function is

$$(5 + 3x + x^2 - x^3)/(1 - 2x - 2x^2 - 2x^3 + x^4), \tag{2.1}$$

while for 2 mod 4 pillows it is

$$(5 + 6x + 3x^2 - 2x^3)/(1 - 2x - 2x^2 - 2x^3 + x^4). \tag{2.2}$$

See Appendix A for a table of values.

Aztec pillows are also the next natural region when considering height functions, since the height function increases as you climb each step. (See Figure 2.2.)

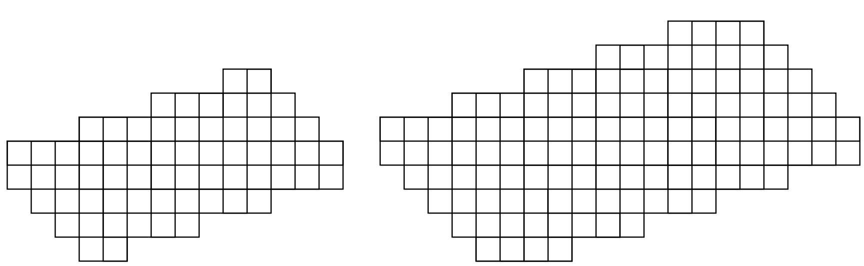

Figure 2.1: Two Aztec Pillows

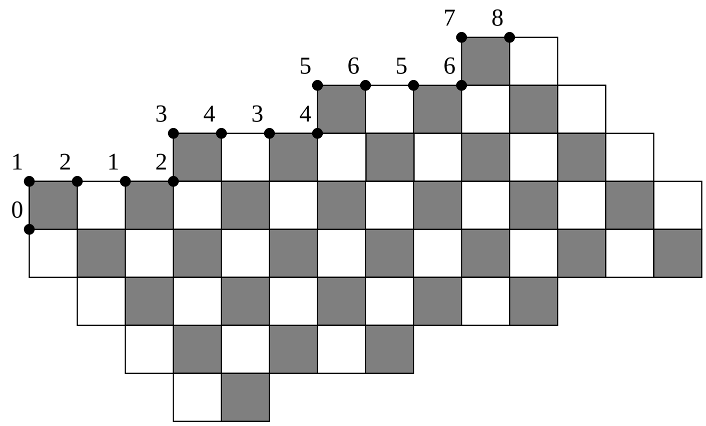

Figure 2.2: The Height Function for an Aztec Pillow

2.2 A Generalization

After exploring a while, I realized that these 0 mod 4 and 2 mod 4 pillows are just a special case of a larger family of pillows, and that maybe writing 0 mod 4 and 2 mod 4 did not capture the information necessary to explain the pillows completely. In particular, taking the height function as a clue, why not allow a pillow to be any region whose height function increases with each step it takes. This means that every step will be composed of following a square up and an odd number of steps to the right, and we should make this object rotationally symmetric (by a rotation of 180 degrees). For the moment, we only consider stepping down from the upper plateau by single steps.

To take into account all of this information, we can reference this object by a vector of the odd step lengths and an extra index to clarify the length of the vector. We subtract 1 from the length of the top plateau so that the last entry in the vector is odd as well. For example, the 0 mod 4 pillows of order n are of the form $(3, \ldots, 3, 3)_n$, while the 2 mod 4 pillows are written $(3, \ldots, 3, 1)_n$. We will call these two nice cases 3-pillows, as we will call vectors of the form $(5, \ldots, 5, i)_n$, where $i \in \{1, 3, 5\}$, the 5-pillows, and define other odd-pillows similarly. Through experimental calculations using vaxmacs and Maple, it appears that 3-, 5-, and 7-pillows all share the form of a smaller number times the square of a larger number, but as of yet I have not found a linear recurrence for these values of degree 10 or smaller.

This more general definition of an Aztec pillow (with arbitrary odd vectors) leads to the broad aim of my research:

Find a general theory of Aztec pillows.

What does this mean? Well, we would like to understand fully the number of tilings of these regions, searching for an explicit structure of such a value. As the reader will see

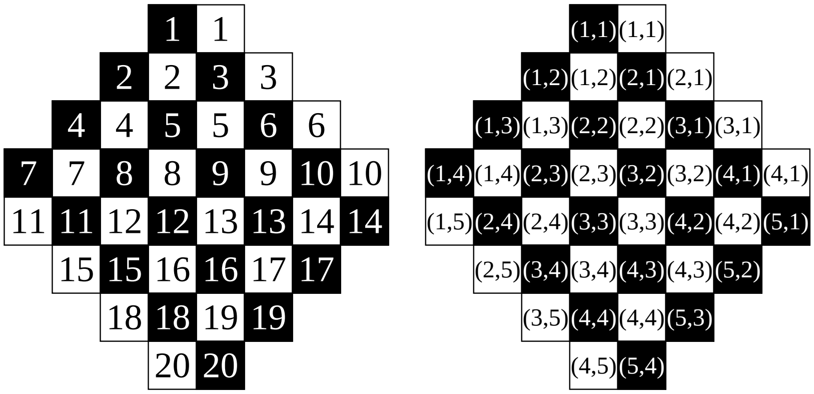

Figure 2.3: Square Positions and Coordinates

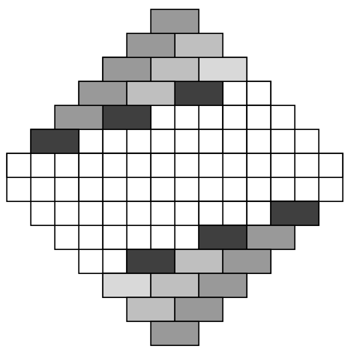

Figure 2.4: Creating an Aztec Pillow from an Aztec Diamond

in Section 3.3, I think the structure is much stronger than just some smaller number times some larger number squared.

2.3 Useful Constructions

Along the way, we need some tools. In particular, we need a coordinate system to be able to refer to specific blocks in the pillow. For an illustrating example, consider the Aztec diamond of order 4 in Figure 1.3. We will label and give coordinates to both the black and white squares as shown in Figure 2.3. In general, for the Aztec diamond of size n, there are $n(n+1)$ squares of each color. The coordinates (x, y) of white squares satisfy all values $1 \leq x \leq n+1$, and $1 \leq y \leq n$; whereas the coordinates (x', y') of black squares satisfy all values $1 \leq x' \leq n$ and $1 \leq y' \leq n+1$.

Giving coordinates to an Aztec diamond is useful because placing dominoes in Aztec diamonds yields Aztec pillows. One needs not place many dominoes to arrive at a pillow. For example, in Figure 2.4, we see how to create the pillow from Figure 2.1(a) using AZ_7 and 3 pairs of dominoes (the ones that are darkly shaded). The lightly shaded dominoes are the dominoes that are forced by the placement of the darker dominoes and thus would be included in any tiling of AZ_7 that includes the darker dominoes.

In general, if n is odd, you need $n-1$ dominoes to restrict AZ_n to a 2 mod 4 3-pillow of order $(n-1)/2$, and if n is even, you need n dominoes to restrict AZ_n to a 0 mod 4 3-pillow of order $n/2$.

Another tool that we will use are the binary Krawtchouk polynomials. (See [16, 20].) A Krawtchouk polynomial $\mathrm{kr}(j,n,k)$ is the coefficient of x^j in the polynomial expression $(1-x)^k(1+x)^{n-k}$.

Some useful manipulations that Krawtchouk polynomials satisfy include:

$$\mathrm{kr}(a,n,c) = (-1)^c\,\mathrm{kr}(n-a,n,c), \tag{2.3}$$

$$\mathrm{kr}(a,n,c) = (-1)^a\,\mathrm{kr}(a,n,n-c), \tag{2.4}$$

which are useful for symmetric arguments,

$$\mathrm{kr}(a,n,c) = \mathrm{kr}(a-1,n-1,c) + \mathrm{kr}(a,n-1,c), \tag{2.5}$$

which is useful for recurrences, and

$$\sum_{i=0}^{n} \mathrm{kr}(a,n,i)\,\mathrm{kr}(i,n,b) = \delta_{ab}2^n, \tag{2.6}$$

which exhibits one of their orthogonality properties. These are all taken from [16].

Chapter 3

PRELIMINARY RESULTS

Now I have my quest. But how to go about it? Up to now, experimental combinatorics has been useful everywhere! I first use vaxmacs and MAPLE to come up with some data, then I use The On-Line Encyclopedia of Integer Sequences [28] to come up with some conjectures, and then I need some way to prove them. Since this is a counting question, I would hope to answer it in a purely combinatorial way; I present the limited results in this vein in Section 3.1. Next, I made use of Harald Helfgott's Theorem [12] (see Equations 3.2 and 3.3) to prove a family of results, and it continues to be useful daily.

3.1 A Combinatorial Approach to $\#(1, \ldots, 1, 3)_n$

I tried to approach this problem in a purely combinatorial way. When I did so, I didn't get too far. I was able to quickly prove a formula for $\#(1, \ldots, 1, 3)_n$, but the $\#(1, \ldots, 1, 3, 1)_n$ case was elusive, and any further case seemed unlikely, but my approach may need to be retooled.

If you consider the region $R = (1, \ldots, 1, 3)_{n-1}$, you can think of it as the Aztec diamond AZ_n with a horizontal domino forced in the top and bottom positions. There is a nice combinatorial argument to prove the formula for $\#R$.

Consider tiling AZ_n in any way. You can do this in $2^{n(n+1)/2}$ ways, by Equation 1.1.

On the other hand, you could break down the number of tilings of AZ_n into cases. You could place a horizontal domino in both the top and bottom rows. You can do this in $\#R$ ways. Otherwise, there would be some vertical tile in either the top or the bottom rows. By Inclusion-Exclusion, we can count the number of ways to tile AZ_n with some vertical tile in the top row, add the number of ways to tile with some vertical tile in the bottom

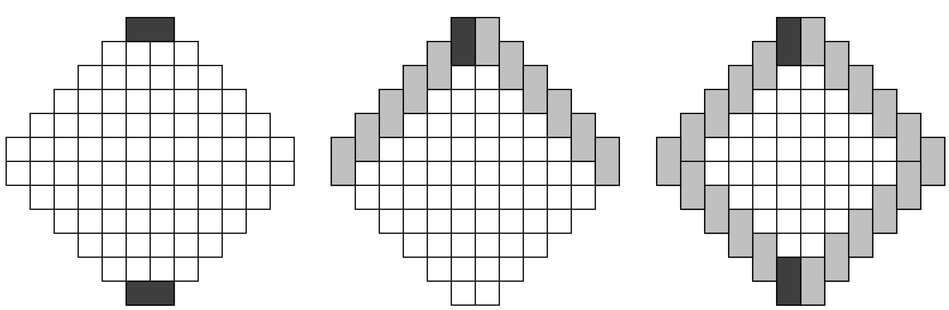

Figure 3.1: Combinatorial Proof of $\#(1, \ldots, 1, 3)_n$ Formula

row (each produces an AZ_{n-1}), and subtract the overcounting of the cases when you place vertical dominoes in both the top and the bottom rows of AZ_n, which results in an AZ_{n-2}. (See Figure 3.1.)

This implies that

$$\#(1,\ldots,1,3)_{n-1} = 2^{n(n+1)/2} - 2\cdot 2^{(n-1)n/2} + 2^{(n-2)(n-1)/2} = 2^{(n-1)(n-2)/2}\left(2^{2n-1} - 2^n + 1\right). \tag{3.1}$$

Trying to use the same approach for $(1,\ldots,1,3,1)_{n-1}$ runs into problems because the resulting regions do not decompose into Aztec diamonds.

3.2 *Helfgott's Theorem*

Another way to proceed has a linear algebraic flavor to it.

Given a subregion S of a tiling region R such that the complement $R\setminus S$ of S in R is tilable, Rick Kenyon wrote an article that explained how to calculate the ratio of $\#(R\setminus S)/\#R$ [15]. In particular, S must be made up of an equal number of black and white vertices. The basic idea is that a certain subdeterminant of the inverse of a Kasteleyn matrix gives this ratio, so if you know how to calculate the entries of this matrix, you can calculate the ratio $\#S/\#R$ with relatively few calculations.

In his Bachelor thesis [12], Harald Helfgott calculated what these matrix entries are for when R is the Aztec diamond. In this way, using the coordinate system from Figure 2.3, he proved that there is a formula involving a determinant for the ratio of the number of tilings of the restricted region over the number of tilings of the Aztec diamond. He gives the following result:

> The probability of a pattern covering white squares $v_1,\ldots,v_k$, and black squares $w_1,\ldots,w_k$ of an Aztec diamond of order n is equal to the absolute value of $|c(v_i,w_j)|_{i,j=1,\ldots,k}$. The coupling function $c(v,w)$ at white square v and black square w is
>
> $$2^{-n}\sum_{j=0}^{x_i-1}\mathrm{kr}(j,n,y_i-1)\,\mathrm{kr}(y_i'-1,n-1,n-(j+x_i'-x_i)) \tag{3.2}$$
>
> for $x_i' > x_i$ and
>
> $$-2^{-n}\sum_{j=x_i}^{n}\mathrm{kr}(j,n,y_i-1)\,\mathrm{kr}(y_i'-1,n-1,n-(j+x_i'-x_i)) \tag{3.3}$$
>
> for $x_i' \le x_i$, where (x_i,y_i) and (x_i',y_i') are the coordinates of v and w in the coordinate system in Figure 2.3, and $\mathrm{kr}(j,n,k)$ is the Krawtchouk polynomial.

This coupling function is hard to work with but it has some nice properties. The coupling matrix that arises from the placement of dominoes to form an Aztec pillow has a particularly nice form.

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \\ -a_{18} & a_{17} & -a_{16} & a_{15} & -a_{14} & a_{13} \\ a_{12} & -a_{11} & a_{10} & -a_9 & a_8 & -a_7 \\ -a_6 & a_5 & -a_4 & a_3 & -a_2 & a_1 \end{pmatrix}$$

Figure 3.2: A Positively Rotationally Sign Alternating Matrix

3.3 Rotationally Sign Alternating Matrices

First, a definition:

Definition: If an $n \times n$ matrix $(a_{i,j})$ has entries that satisfy $a_{i,j} = (-1)^{i+j} a_{n+1-i,n+1-j}$, we call this a **positively rotationally sign alternating** matrix. If $a_{i,j} = (-1)^{i+j+1} a_{n+1-i,n+1-j}$, we call this a **negatively rotationally sign alternating** matrix. In either case, these matrices are called **rotationally sign alternating**.

For an example of such a matrix, see Figure 3.2. Another characterization of rotationally sign alternating matrices is that a matrix M is of this form if and only if $HMH = \pm M$, where H is the matrix with alternating 1's and -1's along the cross diagonal. These matrices are a subclass of the generalized centrosymmetric K-matrices studied in [1, 30], and are related to centrosymmetric matrices introduced in [21] and widely discussed (see e.g. [32]).

Theorem 1. *The coupling matrix of an Aztec pillow is rotationally sign alternating.*

Proof: An Aztec pillow can be derived from an Aztec diamond by a placement of dominoes that is symmetric with respect to a rotation by 180 degrees. In essence, this means that for every white square in position $v_1 = (x_1, y_1)$ and black square in position $w_1 = (x'_1, y'_1)$ in the coordinates of Figure 2.3, there is a white square in position $v_2 = (n+1-x_1, n+2-y_1)$ and a black square in position $w_2 = (n+2-x'_1, n+1-y'_1)$. (Note that we do not require v_1 and w_1 to be in the same domino.)

Given this relation, we can calculate the relationship between $c(v_1, w_1)$ and $c(v_2, w_2)$. Without loss of generality, assume that $x'_1 \leq x_1$ (or else switch the v_i's and w_i's). This implies that $-x_1 \leq -x'_1$, which implies that $x_2 = n+1-x_1 < n+2-x'_1 = x'_2$, so we know which of Equations 3.2 or 3.3 we need to apply in the various cases.

Making extensive use of Equations 2.3 and 2.4, we have

$$
\begin{aligned}
c(v_2,w_2) &= 2^{-n}\sum_{j=0}^{n-x_1} \mathrm{kr}(j,n,n+1-y_1)\,\mathrm{kr}(n-y_1',n-1,n-(j+(n+2-x_1')-(n+1-x_1)))\\
&= 2^{-n}\sum_{j=0}^{n-x_1} \mathrm{kr}(j,n,n-(y_1-1))\,\mathrm{kr}(n-1-(y_1'-1),n-1,n-1-(j-x_1'+x_1))\\
&= 2^{-n}\sum_{j=x_1}^{n} \mathrm{kr}(n-j,n,n-(y_1-1))\,\mathrm{kr}(n-1-(y_1'-1),n-1,n-1-(n-(j+x_1'-x_1)))\\
&= 2^{-n}\sum_{j=x_1}^{n} (-1)^{n-(y_1-1)}(-1)^{j}\,\mathrm{kr}(j,n,y_1-1)\,\cdot\\
&\qquad\qquad (-1)^{n-1-(y_1'-1)}(-1)^{n-(j+x_1'-x_1)}\,\mathrm{kr}(y_1'-1,n-1,n-(j+x_1'-x_1))\\
&= (-1)^{n+x_1+x_1'+y_1+y_1'}\left(-2^{-n}\sum_{j=x_1}^{n}\mathrm{kr}(j,n,y_1-1)\,\mathrm{kr}(y_1'-1,n-1,n-(j+x_1'-x_1))\right)\\
&= (-1)^{n+x_1+x_1'+y_1+y_1'}c(v_1,w_1). \qquad (3.4)
\end{aligned}
$$

Notice that for $v_i=(x_i,y_i)$ and $w_j=(x_j',y_j')$, we have that

$$(-1)^{x_i+y_i+x_j'+y_j'} = -1\cdot(-1)^{x_i+y_i+x_{2d+1-j}'+y_{2d+1-j}'}$$

and therefore for fixed v_i, the sequence $Q(j)=(-1)^{x_i+y_i+x_j'+y_j'}$ is antisymmetric, in that $Q(j)=-Q(2d+1-j)$. Therefore we can relabel the vertices w_j such that the sequence $Q(j)$ becomes $(1,-1,\ldots,1,-1)$ by exchanging w_j and w_{2d+1-j} if necessary. Relabeling the vertices v_i in the same way gives us that $(-1)^{x_i+y_i}=(-1)^i$ and $(-1)^{x_j'+y_j'}=(-1)^j$. This means that Equation 3.4 implies that the entries in the newly indexed coupling matrix satisfy $a_{2d+1-i,2d+1-j}=(-1)^{n+i+j}a_{i,j}$, implying that the coupling matrix is rotationally sign alternating, being either positive or negative depending on the value of n. •

Since we now know that coupling matrices are rotationally sign alternating $2k\times 2k$ matrices, we can see if they have any nice properties. Using Maple, I determined that the absolute value of the determinant of the 2×2, 4×4, 6×6, and 8×8 matrices all decompose nicely into the sum of two squares, and the pieces that are squared are sums of $k\times k$ submatrices. In addition, both types of rotationally alternating sign matrices decompose in this fashion because they differ only by a factor of -1 in each of the last k rows. We have a predictable formula for this sum of two squares.

Here is a construction to be able to explain the formula. We define a set of k-member subsets of $[2k]$. Take a subset I of $[k]$. Create $\widetilde{I}$ by taking $I\cup I'$, where $i\in I'$ if $2k+1-i\in[k]\setminus I$. In this way, each $\widetilde{I}$ has k members. We will define the sets A, A', B, and B' of subsets of $[2k]$. If I has ℓ elements,

$$\text{place } \widetilde{I} \text{ into set } \begin{cases} A & \text{if } \ell \equiv 0 \mod 4, \\ B & \text{if } \ell \equiv 1 \mod 4, \\ A' & \text{if } \ell \equiv 2 \mod 4, \text{ or} \\ B' & \text{if } \ell \equiv 3 \mod 4. \end{cases}$$

Given a $2k \times 2k$ matrix N, we define $M(\widetilde{I})$ to be the $k \times k$ submatrix of N with columns restricted to $j \in \widetilde{I}$, and rows restricted to the first k rows of N.

Conjecture 1. *The formula for the determinant of a positively rotationally sign alternating matrix R is*

$$\det R = \left[\sum_{\widetilde{I} \in A} \det(M(\widetilde{I})) - \sum_{\widetilde{I} \in A'} \det(M(\widetilde{I})) \right]^2 + \left[\sum_{\widetilde{I} \in B} \det(M(\widetilde{I})) - \sum_{\widetilde{I} \in B'} \det(M(\widetilde{I})) \right]^2 . \quad (3.5)$$

As an example, the 4×4 matrix formula is as follows:

$$\det \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ h & -g & f & -e \\ -d & c & -b & a \end{pmatrix} = \left[\det \begin{pmatrix} c & d \\ g & h \end{pmatrix} - \det \begin{pmatrix} a & b \\ e & f \end{pmatrix} \right]^2 + \left[\det \begin{pmatrix} a & c \\ e & g \end{pmatrix} + \det \begin{pmatrix} b & d \\ f & h \end{pmatrix} \right]^2 . \quad (3.6)$$

Note that up to sign, the same formula holds for negatively rotationally sign alternating matrices.

Question 1. *Can I prove Conjecture 1?*

Question 2. *In general, what more can we say about rotationally sign alternating matrices?*

Of course, we are not only concerned with the general case, since we want to look at the specific case of Aztec pillows. When we look at each of the two subdeterminant sums, we see that they each have many factors in common. If we go back to the reason why Aztec pillows were considered "nice", we remember that the number of tilings of Aztec pillows was a small number times a large number squared ($S \times B^2$). It appears that B is a factor of each of the two subdeterminant sums, which would go a long way in explaining the formula that Propp wanted to have understood in [25]. So a logical question is:

Question 3. *In the specific case of any Aztec pillow AP, does the rotationally nice matrix associated with AP help us to understand the value of $\#AP$?*

3.4 Calculating $\#(1,\ldots,1,3)_n$ using Helfgott's Theorem

The simplest application of Helfgott's theorem is when there are only two dominoes restricting positions in the Aztec diamond. One example of those cases is when the dominoes are placed at the very top and bottom of AZ_n. In this case, we find the region $(1,\ldots,1,3)_n$, of which we calculated the number of tilings in Section 3.1. We can now verify this formula independently with the matrix method.

We calculate $(1,\ldots,1,3)_{n-1}$. The coordinates from Figure 2.3 of the white and black squares covered by the dominoes (cells numbered 1 and $n(n+1)$ in AZ_n) are:

$$\begin{array}{ll} (x_1,y_1)=(1,1) & (x'_1,y'_1)=(1,1) \\ (x_2,y_2)=(n,n+1) & (x'_2,y'_2)=(n+1,n) \end{array}$$

This implies that the formulas for calculating the values $c(v_i,w_j)$ are as follows:

For $c(v_1,w_1)$, we note $x'_1 \leq x_1$, so we apply 3.3:

$$\begin{aligned} c(v_1,w_1) &= -2^{-n}\sum_{j=1}^{n} \mathrm{kr}(j,n,0)\,\mathrm{kr}(0,n-1,n-j) \\ &= -2^{-n}\sum_{j=1}^{n}\binom{n}{j}\cdot 1 \\ &= -2^{-n}(2^n-1) \end{aligned}$$

For $c(v_1,w_2)$, we note $x'_2 > x_1$, so we apply 3.2:

$$\begin{aligned} c(v_1,w_2) &= 2^{-n}\sum_{j=0}^{0} \mathrm{kr}(j,n,0)\,\mathrm{kr}(n-1,n-1,-j) \\ &= 2^{-n}\,\mathrm{kr}(0,n,0)\,\mathrm{kr}(n-1,n-1,0) \\ &= 2^{-n}(1\cdot 1) \end{aligned}$$

We can now use Equation 3.4 to establish that the determinant of the Helfgott matrix for $(1,\ldots,1,3)_{n-1}$ is

$$\begin{vmatrix} -2^{-n}(2^n-1) & 2^{-n} \\ -2^{-n}(-1)^n & 2^{-n}(-1)^{n-1}(2^n-1) \end{vmatrix} = 2^{-2n}\left[(2^n-1)^2+1\right]. \tag{3.7}$$

This will be the probability that the dominoes are placed in the top and bottom positions of the Aztec diamond. This was to be expected because we calculated combinatorially (in Section 3.1) the number of tilings of $(1,1,\ldots,1,3)_{n-1}$ to be $2^{(n-1)(n-2)/2}(2^{2n-1}-2^n+1)$. With this information, we expect the above probability to be

$$\frac{\#(1,1,\ldots,1,3)_{n-1}}{\#AZ_n} = \frac{2^{(n-1)(n-2)/2}(2^{2n-1}-2^n+1)}{2^{n(n+1)/2}} \tag{3.8}$$

$$= \frac{2^{n(n+1)/2}2^{-2n}(2^{2n}-2^{n+1}+2)}{2^{n(n+1)/2}} \tag{3.9}$$

$$= 2^{-2n}\left[(2^n-1)^2+1\right]. \tag{3.10}$$

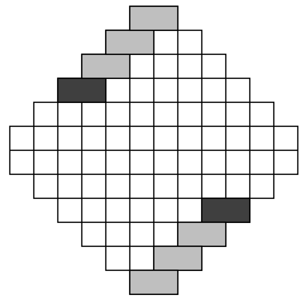

Figure 3.3: A $(1,3,1,1,1)_5$ pillow from AZ_6

3.5 *Calculating* $\#(1,\ldots,1,3,1,\overset{\ell}{\ldots},1)_n$ *Using Helfgott's Theorem*

Another region that is determined by the placement of only two dominoes is a pillow of the form $(1,\ldots,1,3,1,\overset{\ell}{\ldots},1)_n$. Figure 3.3 shows an example where $n=5$ and $\ell=3$.

To get $\#(1,\ldots,1,3,1,\overset{\ell}{\ldots},1)_{n-1}$ from AZ_n, we put two dominoes on the board, in positions $(\ell)(\ell-1)/2+1$ and $n(n+1)/2-\ell(\ell-1)/2$. Calculating the coupling functions,

$$\begin{aligned} c(v_1,w_2) &= 2^{-n}\sum_{j=0}^{0} \mathrm{kr}(j,n,\ell)\,\mathrm{kr}(n-1-\ell,n-1,-j) \\ &= 2^{-n}\,\mathrm{kr}(0,n,\ell)\,\mathrm{kr}(n-1-\ell,n-1,0) \\ &= 2^{-n}\binom{n-1}{n-1-\ell} \\ &= 2^{-n}\binom{n-1}{\ell}. \end{aligned}$$

The calculation of $c(v_2,w_2)$ presents a more difficult challenge. On the next page, we show $c(v_2,w_2)=2^n-\binom{n+1}{\ell}-\cdots-(n+1)-1$.

$$
\begin{aligned}
c(v_2, w_2) &= 2^{-n} \sum_{j=0}^{n-1} \mathrm{kr}(j, n, n-\ell)\, \mathrm{kr}(n-1-\ell, n-1, n-1-j) \\
&= 2^{-n} \sum_{j=0}^{n-1} (-1)^j\, \mathrm{kr}(j, n, \ell)(-1)^{n-1-\ell}(-1)^j\, \mathrm{kr}(\ell, n-1, j) \\
&= 2^{-n}(-1)^{n-1-\ell} \sum_{j=0}^{n-1} \mathrm{kr}(j, n, \ell)\,[\,\mathrm{kr}(\ell, n, j) - \mathrm{kr}(\ell-1, n-1, j)] \\
&= 2^{-n}(-1)^{n-1-\ell} \sum_{j=0}^{n-1} \mathrm{kr}(j, n, \ell)\,[\,\mathrm{kr}(\ell, n, j) - \mathrm{kr}(\ell-1, n, j) + \mathrm{kr}(\ell-2, n-1, j)] \\
&= 2^{-n}(-1)^{n-1-\ell} \sum_{j=0}^{n-1} \mathrm{kr}(j, n, \ell) \left[\sum_{k=1}^{\ell} (-1)^{\ell-k}\, \mathrm{kr}(k, n, j) + (-1)^{\ell}\, \mathrm{kr}(0, n-1, j) \right] \\
&= 2^{-n}(-1)^{n-1-\ell} \sum_{j=0}^{n-1} \mathrm{kr}(j, n, \ell) \left[\sum_{k=0}^{\ell} (-1)^{\ell-k}\, \mathrm{kr}(k, n, j) \right] \\
&= 2^{-n}(-1)^{n-1-\ell} \sum_{k=0}^{\ell} (-1)^{\ell-k} \sum_{j=0}^{n-1} \mathrm{kr}(k, n, j)\, \mathrm{kr}(j, n, \ell) \\
&= 2^{-n}(-1)^{n-1-\ell} \sum_{k=0}^{\ell} (-1)^{\ell-k} \left(\sum_{j=0}^{n} \mathrm{kr}(k, n, j)\, \mathrm{kr}(j, n, \ell) - \mathrm{kr}(k, n, n)\, \mathrm{kr}(n, n, \ell) \right) \\
&= 2^{-n}(-1)^{n-1-\ell} \sum_{k=0}^{\ell} (-1)^{\ell-k} \left(\sum_{j=0}^{n} \mathrm{kr}(k, n, j)\, \mathrm{kr}(j, n, \ell) - (-1)^k \binom{n}{k} (-1)^{\ell} \right) \\
&= 2^{-n}(-1)^{n-1-\ell} \sum_{k=0}^{\ell} \left((-1)^{\ell-k} \delta_{k\ell} 2^n - \binom{n}{k} \right) \\
&= 2^{-n}(-1)^{n-1-\ell} \left[2^n - \sum_{k=0}^{\ell} \binom{n}{k} \right] \\
&= 2^{-n}(-1)^{n-1-\ell} \left[\sum_{k=\ell+1}^{n} \binom{n}{k} \right].
\end{aligned}
$$

The determinant now gives

$$
2^{-2n} \left[\left(2^n - \binom{n}{\ell} - \cdots - (n) - 1 \right)^2 + \binom{n-1}{k}^2 \right]
$$

$$
= 2^{-2n} \left[\left(\sum_{j=\ell+1}^{n} \binom{n}{j} \right)^2 + \binom{n-1}{\ell}^2 \right]
$$

This is be the probability that the dominoes are placed in the correct positions of the Aztec diamond, so multiplying by $2^{n(n+1)/2}$ gives

$$\#(1,\ldots,1,3,1,\overset{\ell}{\ldots},1)_{n-1} = 2^{(n-1)(n-2)/2}\frac{1}{2}\left[\left(\sum_{j=\ell+1}^{n}\binom{n}{j}\right)^2 + \binom{n-1}{\ell}^2\right]. \tag{3.11}$$

Note the form is a sum of squares. This formula looks very combinatorial, so I would hope to be able to answer the following question:

Question 4. *Can we prove Equation 3.11 by purely combinatorial means?*

Appendix B contains a table of values for $(1,\ldots,1,3,1,\ldots,1)_n$ for different values of n and ℓ.

3.6 Some Specific Cases

Note that this reduces to the result from Section 3.4 when $\ell = 0$. Another region that has a nice formula experimentally is $(3,1\ldots,1)_n$. This is the subcase of the previous section when $\ell = n-1$. When we apply Equation 3.11, we obtain the following verification of the experimental result.

$$\#(3,1,\ldots,1)_n = 2^{n(n-1)/2}\left[\frac{(n+1+1)^2+(n)^2}{2}\right] = 2^{n(n-1)/2}\left[(n+1)^2+1\right]. \tag{3.12}$$

Again, this formula seems to lend itself to a nice combinatorial proof, but none has been found to date.

Chapter 4

FUTURE WORK

In addition to the questions asked in the previous chapters, other questions arise in the search for a greater "pillow theory". I present them in this chapter, along with the limited progress that has been accomplished for each.

4.1 Additional Questions

In Section 2.1, we are introduced to the generating functions of 3-pillows "small number" parts. In Section 3.3, we notice that the formula should be a sum of two squares. This leads to the next question.

Question 5. *Is there a way to prove algebraically that coefficients of the two generating functions are sums of squares?*

The pattern of the prime factors in the coefficients of the expansion of the generating functions leads me to think that there should be algebraic way to answer this question. We may be able to use the criterion for numbers to be sums of squares — that every prime divisor congruent to 3 mod 4 must be of even power in the prime decomposition of the number. This has not yet led to fruition. In addition, the denominator is of a nice form when we take it mod p prime. Maybe I can exploit this?

In Chapter 3, I concerned myself only with pillow vectors with entries equal to either 1 or 3. However, pillow theory allows for vectors with arbitrary odd constants. This leads to some questions. We see that odd pillows are created from successive Aztec diamonds by introducing regular dominoes. Each 3-pillow independently arises from a unique Aztec diamond.

Question 6. *Should we really consider 2 mod 4 pillows and 0 mod 4 pillows as separate sequences? Or should we consider them as one interleaved sequence?*

This same question applies to sequences of any odd pillow.

We have a nice pattern for the small parts of 3-pillows.

Question 7. *Is there a pattern for the small parts of all odd pillows?*

I have calculated many values for 5-pillows and many patterns abound; they have yet to lead to a concrete formula. See Appendix C for a subset of the data. This question had a recent interesting turn. Using Equation 3.5, I calculated the values for the two squares that are summed. Sometimes the two squares summed to the coefficient in the power series equations Equations 2.1 and 2.2. Other times they summed to twice that number. This leads me to infer that the coefficients of the power series may be simplest way to express the values, but may not be the most natural. Stay tuned ...

Using Helfgott's matrix method worked well to calculate $\#(1,\ldots,1,3,1,\ldots,1)_n$, but I only tried it in this case. Maybe the calculations for more regions will give more insight?

Question 8. *Does Helfgott's Theorem give a nice form for* $\#(3,1,\ldots,1,3,1,\ldots,1)_n$*?*

I choose this as the next value to be computed because calculating for four dominoes is not hard, and I envision being able to build a 3-pillow step-by-step by placing one domino after another to restrict AZ_n.

In his 1994 article, Jockusch uses the rotational symmetry of a region to prove some results having to do with sums of squares without explaining the structure of the squares in themselves. Perhaps I will prove something similar for pillows.

Question 9. *How does Jockusch's article apply to pillows?*

In addition, the study of binary Krawtchouk polynomials seems intriguing, and in my trials to find formulas for the number of tilings of Aztec pillows, I have found one seemingly new identity. It would be exciting to find some more.

Question 10. *Can I find new Krawtchouk identities that will help in Pillow Theory?*

A question that people ask about Aztec diamonds and other regions is "what does a random tiling of the region look like"? This led to a discovery of polar regions in Aztec diamonds and hexagonal tilings. This strays from the enumeration aspect of my research, but is an interesting question nonetheless. See Figure 4.1 for an example.

Question 11. *What does a randomly chosen tiling of an Aztec pillow look like?*

With all these questions, I aim to come up with a whole Theory of Pillows, a step forward in understanding which regions have tilings that are easily enumerable and which do not.

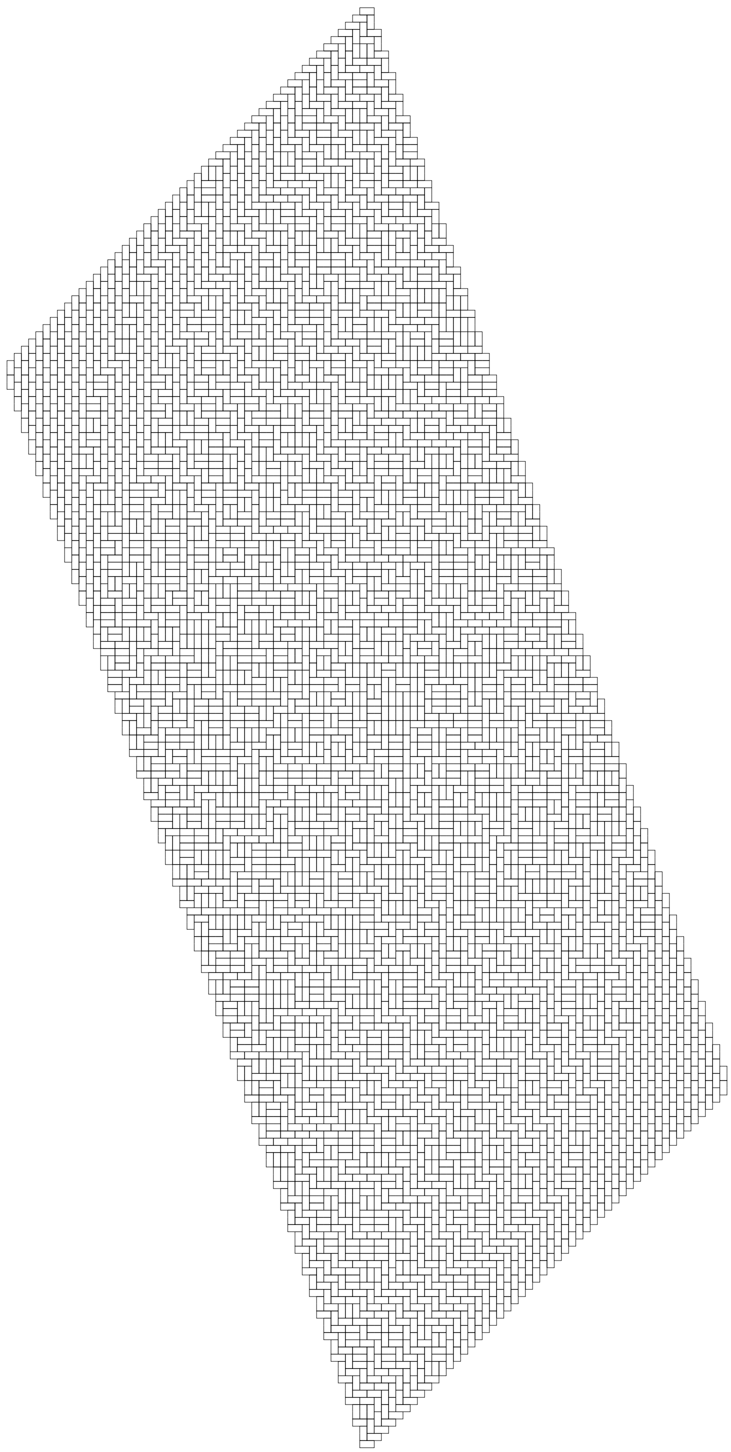

Figure 4.1: A Randomly Tiled $(3, \ldots, 3)_{50}$

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Appendix A

TABLE OF 3-PILLOWS

Included in these tables are the number of tilings of the two families of 3-pillows divided by an appropriate power of two. For 0 mod 4 3-pillows, we have divided by $2^{(2n)(2n-1)/2}$, and for 2 mod 4 3-pillows, we have divided by $2^{(2n-2)(2n-3)/2-1}$. The non-square part was calculated using a product of the bold faced terms (or some factors thereof).

A.1 Table of 2 mod 4 3-pillows ($\#(3,\ldots,3,1)_n$)

n	$\#(3,\ldots,3,1)_n$	Non-Square Part
1	$\mathbf{2}$	2
2	$2^2 \cdot \mathbf{5}$	5
3	$\mathbf{2^10}$	16
4	$2^4 \cdot \mathbf{3^2} \cdot \mathbf{5} \cdot 19^2$	45
5	$\mathbf{2^9} \cdot 3^4 \cdot \mathbf{5} \cdot 11^2 \cdot \mathbf{13}$	130
6	$2^6 \cdot 3^2 \cdot \mathbf{13} \cdot \mathbf{29} \cdot 43^2 \cdot 71^2$	377
7	$\mathbf{2^{28}} \cdot 7^2 \cdot \mathbf{17^3} \cdot 31^2$	1088
8	$2^8 \cdot \mathbf{5} \cdot \mathbf{17^3} \cdot 19^2 \cdot \mathbf{37} \cdot 53^2 \cdot 71^2 \cdot 89^2$	3145
9	$\mathbf{2^{17}} \cdot \mathbf{3^2} \cdot \mathbf{5} \cdot 11^2 \cdot 19^4 \cdot 59^2 \cdot 61^2 \cdot \mathbf{101} \cdot 241^2$	9090
10	$2^{10} \cdot 23^2 \cdot 43^2 \cdot \mathbf{109} \cdot \mathbf{241} \cdot 263^2 \cdot 439^2 \cdot 461^2 \cdot 593^2$	26269
11	$\mathbf{2^{34}} \cdot 3^4 \cdot \mathbf{5^3} \cdot 7^4 \cdot 11^2 \cdot \mathbf{13^3} \cdot 19^2 \cdot 23^2 \cdot 47^2 \cdot 71^2 \cdot \mathbf{73} \cdot 167^2$	75920
12	$2^{12} \cdot 3^4 \cdot 5^4 \cdot 79^2 \cdot \mathbf{313} \cdot \mathbf{701} \cdot 911^2 \cdot 1429^2 \cdot 1481^2 \cdot 1741^2 \cdot 3691^2$	219413

A.2 Table of 0 mod 4 3-pillows ($\#(3,\ldots,3,3)_n$)

n	$\#(3,\ldots,3,3)_n$	Non-Square Part
1	$\mathbf{5}$	5
2	$3^2 \cdot \mathbf{13}$	13
3	$19^2 \cdot \mathbf{37}$	37
4	$\mathbf{109} \cdot 263^2$	109
5	$3^4 \cdot \mathbf{313} \cdot 911^2$	313
6	$\mathbf{5} \cdot 11^4 \cdot 31^2 \cdot 151^2 \cdot \mathbf{181}$	905
7	$101^2 \cdot 103^2 \cdot \mathbf{2617} \cdot 8363^2$	2617
8	$31^2 \cdot \mathbf{7561} \cdot 27283^2 \cdot 35149^2$	7561
9	$3^{10} \cdot 5^2 \cdot \mathbf{13} \cdot 29^2 \cdot \mathbf{41^4} \cdot 43^2 \cdot 211^2 \cdot 1723^2$	21853
10	$47^2 \cdot \mathbf{137} \cdot \mathbf{461} \cdot 313949^2 \cdot 8647^2 \cdot 298999^2$	63157
11	$\mathbf{5^{10}} \cdot \mathbf{7^2} \cdot \mathbf{149} \cdot 10399^2 \cdot 39551^2 \cdot 55201^2 \cdot 10099^2$	182525
12	$19^2 \cdot \mathbf{37} \cdot \mathbf{53^3} \cdot 107^2 \cdot \mathbf{269} \cdot 431^2 \cdot 809^2 \cdot 89317^2 \cdot 61723^2 \cdot 5779^2$	527509

Appendix B

TABLE OF 131'S

n	$(3,1's)_n$	$(1,3,1's)_n$	$(1,1,3,1's)_n$	$(1,1,1,3,1's)_n$	$(1,\overset{4}{\ldots},1,3,1's)_n$	$(1,\overset{5}{\ldots},1,3,1's)_n$	$(1,\overset{6}{\ldots},1,3,1's)_n$	$(1,\overset{7}{\ldots},1,3,1's)_n$	$(1,\overset{8}{\ldots},1,3,1's)_n$
1	5								
2	10	25							
3	17	65	113						
4	26	146	346	481					
5	37	292	932	1637	1985				
6	50	533	2248	5013	7218	8065			
7	65	905	4937	13897	24201	30529	32513		
8	82	1450	10018	35218	74530	108970	126034	130561	
9	101	2216	19016	82436	211460	363080	469160	513125	523265
10	122	3257	34112	179972	556040	1126148	1656128	1963193	2072698

Appendix C

TABLE OF 5-PILLOWS

Included in these tables are the number of tilings of the three families of 5-pillows divided by an appropriate power of two. The non-square part was calculated using a product of the bold faced terms (or some factors thereof). "Quotient" represents the quotient between successive terms.

C.1 Table of 2 mod 6 5-pillows ($\#(5,\ldots,5,1)_n$)

n	$\#(5,\ldots,5,1)_n$	Non-Square Part	Quotient
1	$\mathbf{2}$	2	
			$\rangle 6.5$
2	$2^2 \cdot \mathbf{13}$	13	
			$\rangle 7.69$
3	$\mathbf{2^6} \cdot 3^2 \cdot \mathbf{5^2}$	100	
			$\rangle 3.61$
4	$2^4 \cdot 3^2 \cdot \mathbf{19^2} \cdot 29^2$	361	
			$\rangle 8.04$
5	$\mathbf{2^{13}} \cdot 3^6 \cdot 13^2 \cdot \mathbf{1453}$	2906	
			$\rangle 3.6796$
6	$2^6 \cdot 3^4 \cdot \mathbf{17^2} \cdot \mathbf{37} \cdot 98873^2$	10693	
			$\rangle 7.7957$
7	$\mathbf{2^{35}} \cdot \mathbf{5^5} \cdot \mathbf{521} \cdot 6277^2$	83360	
			$\rangle 3.8047$
8	$2^8 \cdot 3^2 \cdot \mathbf{5} \cdot \mathbf{229} \cdot \mathbf{277} \cdot 5273^2 \cdot 69689357^2$	317165	
			$\rangle 7.6562$
9	$\mathbf{2^{21}} \cdot 11^2 \cdot 31^6 \cdot \mathbf{97} \cdot 1789^2 \cdot \mathbf{12517} \cdot 235723^2$	2428298	
			$\rangle 3.8303$
10	$2^{10} \cdot 13^2 \cdot 127^2 \cdot \mathbf{9301217} \cdot 100799836548408841^2$	9301217	
			$\rangle 7.6435$
11	$\mathbf{2^{44}} \cdot \mathbf{5} \cdot 11^2 \cdot \mathbf{149} \cdot \mathbf{23857} \cdot 1185523^2 \cdot 1573399^2 \cdot 10939021^2$	71093860	
			$\rangle 3.82888$
12	$2^{12} \cdot \mathbf{5^3} \cdot \mathbf{54442097} \cdot 80619627749^2 \cdot 7559882680695003557^2$	272210485	

C.2 Table of 4 mod 6 5-pillows ($\#(5,\ldots,5,3)_n$)

n	$\#(5,\ldots,5,3)_n$	Non-Square Part	Quotient
1	$\mathbf{5}$	5	
			$\rangle 5.8$
2	$3^2 \cdot \mathbf{29}$	29	
			$\rangle 4.48$
3	$\mathbf{2} \cdot \mathbf{5^3} \cdot 7^2 \cdot \mathbf{13}$	130	
			$\rangle 6.7769$
4	$3^6 \cdot 41^2 \cdot \mathbf{881}$	881	
			$\rangle 4.0374$
5	$2^2 \cdot 79^2 \cdot 953^2 \cdot \mathbf{3557}$	3557	
			$\rangle 7.5605$
6	$\mathbf{26893} \cdot 49644383^2$	26893	
			$\rangle 3.8225$
7	$\mathbf{2^{10}} \cdot \mathbf{5^2} \cdot 37^2 \cdot 257 \cdot 258007163^2$	102800	
			$\rangle 7.6851$
8	$3^6 \cdot \mathbf{829} \cdot \mathbf{953} \cdot 1321^2 \cdot 68947^2 \cdot 111791^2$	790037	
			$\rangle 3.8115$
9	$2^4 \cdot \mathbf{3^2} \cdot \mathbf{5} \cdot 17^2 \cdot 31^2 \cdot \mathbf{61} \cdot \mathbf{1097} \cdot 2211657330256441^2$	3011265	
			$\rangle 7.6688$
10	$\mathbf{3^4} \cdot 29^2 \cdot \mathbf{137} \cdot \mathbf{2081} \cdot 12149633^2 \cdot 524888608699277^2$	23092857	
			$\rangle 3.8227$
11	$\mathbf{2^9} \cdot 11^2 \cdot 59^2 \cdot \mathbf{149^3} \cdot \mathbf{296237} \cdot 7549169^2 \cdot 2951523929123521^2$	88278626	
			$\rangle 7.65407$
12	$\mathbf{3^4} \cdot \mathbf{5} \cdot 19^2 \cdot 109^2 \cdot \mathbf{593} \cdot 3581^2\mathbf{25321} \cdot 201800713^2 \cdot 3356075623404281417^2$	675690885	

C.3 Table of 0 mod 6 5-pillows ($\#(5,\ldots,5,5)_n$)

n	$\#(5,\ldots,5,5)_n$	Non-Square Part	Quotient
1	$\mathbf{13}$	13	
			$\rangle 2.615$
2	$\mathbf{2} \cdot 7^2 \cdot \mathbf{17}$	34	
			$\rangle 8.97$
3	$3^2 \cdot \mathbf{5} \cdot 29^2 \cdot \mathbf{61}$	305	
			$\rangle 3.754$
4	$2^2 \cdot \mathbf{5} \cdot 31^2 \cdot 89^2 \cdot \mathbf{229}$	1145	
			$\rangle 7.571$
5	$19^2 \cdot \mathbf{8669} \cdot 43003^2$	8669	
			$\rangle 3.885$
6	$\mathbf{2^{10}} \cdot \mathbf{5^3} \cdot \mathbf{421} \cdot 14010851^2$	33680	
			$\rangle 7.558$
7	$\mathbf{3^2} \cdot \mathbf{5^3} \cdot 1677^2 \cdot \mathbf{5657} \cdot 107775611^2$	254565	
			$\rangle 3.8506$
8	$2^4 \cdot 3^2 \cdot \mathbf{17} \cdot \mathbf{23^2} \cdot 29^2 \cdot 31^2 \cdot \mathbf{109} \cdot 3593^2 \cdot 178947631^2$	980237	
			$\rangle 7.6346$
9	$\mathbf{13} \cdot 17^2 \cdot \mathbf{575677} \cdot 18231881^2 \cdot 399960749339^2$	7483801	
			$\rangle 3.8278$
10	$\mathbf{2^9} \cdot \mathbf{13} \cdot \mathbf{29} \cdot 113^2 \cdot \mathbf{27993} \cdot 1471746643^2 \cdot 3901319205691^2$	28646722	
			$\rangle 7.65414$
11	$3^2 \cdot \mathbf{5} \cdot \mathbf{149} \cdot 49831^2 \cdot \mathbf{294317} \cdot 7516933^2 \cdot 181512950498412485 3^2$	219266165	
			$\rangle 3.82487$
12	$2^6 \cdot 3^4 \cdot \mathbf{5} \cdot \mathbf{167733113} \cdot 1203004071587^2 \cdot 217164367297797072143^2$	838665565	