Combinatorial interpretations in affine Coxeter groups

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Joint work with Brant C. Jones, James Madison University
What is a Coxeter group?

A **Coxeter group** is a group with

- **Generators:** $S = \{s_1, s_2, \ldots, s_n\}$
- **Relations:**
  - $s_i^2 = 1$, $(s_is_j)^m_{i,j} = 1$ where $m_{i,j} \geq 2$ or $= \infty$
  - $m_{i,j} = 2$: $(s_is_j)(s_is_j) = 1 \rightarrow s_is_j = s_js_i$ (they commute)
  - $m_{i,j} = 3$: $(s_is_j)(s_is_j)(s_is_j) = 1 \rightarrow s_is_js_i = s_js_is_j$ (braid relation)
  - $m_{i,j} = \infty$: $s_i$ and $s_j$ are not related.

Why Coxeter groups?

- They’re awesome.
- Discrete Geometry: Symmetries of regular polyhedra.
- Algebra: Symmetric group generalizations. (Kac-Moody, Hecke)
- Geometry: Classification of Lie groups and Lie algebras
Examples of Coxeter groups

A shorthand notation is the **Coxeter graph**:

- **Vertices**: One for every generator $i$
- **Edges**: Create an edge between $i$ and $j$ when $m_{i,j} \geq 3$
  Label edges with $m_{i,j}$ when $\geq 4$.

### Dihedral group

- **Generators**: $s, t$.
- **Relation**: $(st)^m = 1$.

Symmetry group of regular $m$-gon.
Examples of Coxeter groups

(Finite) \( n \)-permutations \( S_n \)

An \( n \)-permutation is a permutation of \( \{1, 2, \ldots, n\} \), (e.g. \( 214536 \)).

Every \( n \)-permutation is a product of adjacent transpositions.

\( s_i : (i) \leftrightarrow (i + 1) \). (e.g. \( s_4 = 123546 \)).

Example. Write \( 214536 \) as \( s_3s_4s_1 \).

This is a Coxeter group:

- Generators: \( s_1, \ldots, s_{n-1} \)
- \( s_is_j = s_js_i \) when \( |i - j| \geq 2 \) (commutation relation)
- \( s_is_js_i = s_js_is_j \) when \( |i - j| = 1 \) (braid relation)
Examples of Coxeter groups

Affine $n$-Permutations $\tilde{S}_n$

- Generators: $s_0, s_1, \ldots, s_{n-1}$
- Relations:

  - $s_0$ has a braid relation with $s_1$ and $s_{n-1}$
  - How does this impact 1-line notation?
    - Perhaps interchanges 1 and $n$?
    - Not quite! (Would add a relation)
  - Better to view graph as:
    - Every generator is the same.
Examples of Coxeter groups

**Affine \( n \)-Permutations** \( \widetilde{S}_n \)  

Write an element \( \tilde{w} \in \widetilde{S}_n \) in 1-line notation as a permutation of \( \mathbb{Z} \).

Generators transpose **infinitely many** pairs of entries:
\[ s_i : (i) \leftrightarrow (i+1) \ldots (n+i) \leftrightarrow (n+i+1) \ldots (−n+i) \leftrightarrow (−n+i+1) \ldots \]

<table>
<thead>
<tr>
<th>( s_1 )</th>
<th>( \ldots )</th>
<th>(-4)</th>
<th>(-3)</th>
<th>(-1)</th>
<th>( 0 )</th>
<th>( 2 )</th>
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<th>( 10 \ldots )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s_0 )</td>
<td>( \ldots )</td>
<td>(-3)</td>
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<td>(-2)</td>
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<tr>
<td>( s_1 s_0 )</td>
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<td>(-2)</td>
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<td>( 5 )</td>
<td>( 7 )</td>
<td>( 10 )</td>
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</table>

**Symmetry:** Can think of as integers wrapped around a cylinder.

\( \tilde{w} \) is defined by the **window** \( [\tilde{w}(1), \tilde{w}(2), \ldots, \tilde{w}(n)] \).

\( s_1 s_0 = [0, 1, 3, 6] \)
Examples of Coxeter groups

Affine $n$-Permutations $\tilde{S}_n$
Examples of Coxeter groups

**Affine \( n \)-Permutations** \( \tilde{S}_n \) — elements correspond to alcoves.
Properties of Coxeter groups

For a elements $w$ in a Coxeter group $W$,

- $w$ may have multiple expressions.
  - Transfer between them using relations.

**Example.** In $S_4$, $w = s_1 s_2 s_3 s_1 = s_1 s_2 s_1 s_3 = s_2 s_1 s_2 s_3 = s_2 s_1 s_2 s_3 s_1 s_1$

- $w$ has a shortest expression (this length: **Coxeter length**)

For a Coxeter group $\widetilde{W}$,

- An induced subgraph of $\widetilde{W}$’s Coxeter graph is a subgroup $W$
- Every element $\tilde{w} \in \widetilde{W}$ can be written $\tilde{w} = w^0 w$, where $w^0 \in \widetilde{W}/W$ is a coset representative and $w \in W$. 
**Key concept:** View $S_n$ as a subgroup of $\tilde{S}_n$.

- Write $\tilde{w} = w^0 w$, where $w^0 \in \tilde{S}_n/S_n$ and $w \in S_n$.
- $w^0$ determines the entries; $w$ determines their order.

**Example.** For $\tilde{w} = [-11, 20, -3, 4, 11, 0] \in \tilde{S}_6$,

$$w^0 = [-11, -3, 0, 4, 11, 20] \quad \text{and} \quad w = [1, 3, 6, 4, 5, 2].$$

Many interpretations of these *minimal length coset representatives.*
Combinatorial interpretations of $\tilde{S}_n/S_n$

Combinatorial interpretations of $\tilde{S}_n/S_n$

Redacted: $[−4,−3,7,10]$,

Reduced expression: $s_1s_0s_2s_3s_1s_0s_2s_3s_1s_0$

Window notation

Abacus diagram

Elements of $\tilde{S}_n/S_n$

Bounded partition

Core partition

Root lattice point

Bounded partition

Combinatorial interpretations in affine Coxeter groups

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An abacus model for $\tilde{S}_n/S_n$

(James and Kerber, 1981) Given $w^0 = [w_1, \ldots, w_n] \in \tilde{S}_n/S_n$,

- Place integers in $n$ runners.
- Circled: beads. Empty: gaps
- Bijection: Given $w^0$, create an abacus where each runner has a lowest bead at $w_i$.

Example: $[-4, -3, 7, 10]$

These abaci are flush and balanced.

The generators act nicely on the abacus.
Action of generators on the abacus

- $s_i$ acts by interchanging runners $i$ and $i + 1$.
- $s_0$ acts by interchanging runners 1 and $n$, with level shifts.

**Example:** Consider $[-4, -3, 7, 10] = s_1 s_0 s_1 s_3 s_2 s_0 s_3 s_1 s_0$.

Start with $id = [1, 2, 3, 4]$ and apply the generators one by one:

```
[1, 2, 3, 4]  S0  [0, 2, 3, 5]  S1  [0, 1, 3, 6]  S3  [-1, 1, 4, 6]  S0  [-1, 0, 5, 6]
```

```
1  2  3  4  5  6  7  8  9  10 11 12 13 14 15 16
-11 -10 -9 -8 -7 -6 -5 -4 -3 -2 -1 0  9  10  11  12 13 14 15 16
```

```
1  2  3  4  5  6  7  8  9  10 11 12 13 14 15 16
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Combinatorial interpretations of $\tilde{S}_n/S_n$

Elements of $\tilde{S}_n/S_n$

- Window notation
- Reduced expression
- Abacus diagram
- Bounded partition
- Core partition
- Root lattice point

$[-4,-3,7,10]$
Integer partitions and $n$-core partitions

For an integer partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ drawn as a Ferrers diagram,

The **hook length** of a box is $\#$ boxes below and to the right.

An **$n$-core** is a partition with no boxes of hook length dividing $n$.

**Example.** $\lambda$ is a 4-core, 8-core, 11-core, 12-core, etc.
$\lambda$ is NOT a 1-, 2-, 3-, 5-, 6-, 7-, 9-, or 10-core.
Core partitions for $\tilde{S}_n/S_n$

Elements of $\tilde{S}_n/S_n$ are in bijection with $n$-cores.

**Bijection:** $\{\text{abaci}\} \leftrightarrow \{n\text{-cores}\}$

**Rule:** Read the boundary steps of $\lambda$ from the abacus:

- A bead $\leftrightarrow$ vertical step
- A gap $\leftrightarrow$ horizontal step

Fact: Abacus flush with $n$-runners $\leftrightarrow$ partition is $n$-core.
Action of generators on the core partition

- Label the boxes of $\lambda$ with residues.
- $s_i$ acts by adding or removing boxes with residue $i$.

**Example:** Let’s see the *deconstruction* of $s_1 s_0 s_2 s_1 s_3 s_2 s_0 s_3 s_1 s_0$:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
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</table>

Applying generator $s_1$ removes all removable 1-boxes.
Combinatorial interpretations of $\widetilde{S}_n/S_n$

Elements of $\widetilde{S}_n/S_n$

- Reduced expression
- Window notation
- Abacus diagram
- Bounded partition
- Core partition
- Root lattice point

$[-4, -3, 7, 10]$

$s_1 s_0 s_2 s_3 s_1 s_0 s_2 s_3 s_1 s_0$

$(-1, 2, 1, -2)$
Bounded partitions for $\widetilde{S}_n/S_n$

A partition $\beta = (\beta_1, \ldots, \beta_k)$ is \textit{b-bounded} if $\beta_i \leq b$ for all $i$.

Elements of $\widetilde{S}_n/S_n$ are in bijection with $(n-1)$-bounded partitions.

\textbf{Bijection:} (Lapointe, Morse, 2005)

$$\{n\text{-cores } \lambda\} \leftrightarrow \{(n - 1)\text{-bounded partitions } \beta\}$$

- Remove all boxes of $\lambda$ with hook length $\geq n$
- Left-justify remaining boxes.

\[\begin{array}{cccccc}
10 & 9 & 6 & 5 & 2 & 1 \\
7 & 6 & 3 & 2 \\
6 & 5 & 2 & 1 \\
3 & 2 \\
2 & 1 \\
\end{array}\]

$\lambda = (6, 4, 4, 2, 2)$

$\rightarrow$

$\rightarrow$

$\beta = (2, 2, 2, 2, 2)$
Canonical reduced expression for $\tilde{S}_n/S_n$

Given the bounded partition, read off the reduced expression:

**Method:** (Berg, Jones, Vazirani, 2009)

- Fill $\beta$ with residues $i$
- Tally $s_i$ reading right-to-left in rows from bottom-to-top

**Example.** $[-4, -3, 7, 10] = s_1 s_0 s_2 s_1 s_3 s_2 s_0 s_3 s_1 s_0$.

- The Coxeter length of $w$ is the number of boxes in $\beta$. 
Fully commutative elements

**Definition.** An element in a Coxeter group is **fully commutative** if it has only one reduced expression (up to commutation relations).

**Example.** In $S_4$, $s_1s_2s_3s_1$ is **not fully commutative** because

$$s_1s_2s_3s_1 = s_1s_2s_1s_3 \neq s_2s_1s_2s_3$$

**Question:** What is $s_1s_2s_1$ in **1-line** notation?

**Answer:** $3 \ 2 \ 1 \ 4 \ 5 \ 6 \ldots$
Enumerating fully commutative elements

**Question:** How many fully commutative elements are there in $S_n$?

**Answer:** Catalan many!

$S_1$: 1. id

$S_2$: 2. id, $s_1$

$S_3$: 5. id, $s_1$, $s_2$, $s_1s_2$, $s_2s_1$

$S_4$: 14. id, $s_1$, $s_2$, $s_3$, $s_1s_2$, $s_2s_1$, $s_2s_3$, $s_3s_2$, $s_1s_3$, $s_1s_2s_3$, $s_1s_3s_2$, $s_2s_1s_3$, $s_3s_2s_1$, $s_2s_1s_3s_2$

**Key idea:** (Billey, Jockusch, Stanley, 1993)

$w$ is fully commutative $\iff w$ is 321-avoiding.

(Knuth, 1973) These are counted by the Catalan numbers.
Question: How many fully commutative elements are there in $\tilde{S}_n$?

Answer: Infinitely many! (Even in $\tilde{S}_3$.)

$id, s_1, s_1 s_2, s_1 s_2 s_0, s_1 s_2 s_0 s_1, s_1 s_2 s_0 s_1 s_2, \ldots$

Multiplying the generators cyclically does not introduce braids.

This is not the right question.
Enumerating fully commutative elements

**Question:** How many fully commutative elements are there in $\tilde{S}_n$, with Coxeter length $\ell$?

In $\tilde{S}_3$: id, $s_1$, $s_0 s_1$, $s_0 s_2$, $s_0 s_1 s_2$, $s_0 s_1 s_2 s_1$, ...

**Question:** Determine the coefficient of $q^\ell$ in the generating function

$$f_n(q) = \sum_{\tilde{w} \in \tilde{S}_n^{FC}} q^{\ell(w)}.$$  

$$f_3(q) = 1q^0 + 3q^1 + 6q^2 + 6q^3 + \ldots$$

**Answer:** Consult your friendly computer algebra program.
Brant calls up and says: “Hey Chris, look at this data!”

\[ f_3(q) = 1 + 3q + 6q^2 + 6q^3 + 6q^4 + 6q^5 + \cdots \]
\[ f_4(q) = 1 + 4q + 10q^2 + 16q^3 + 18q^4 + 16q^5 + 18q^6 + \cdots \]
\[ f_5(q) = 1 + 5q + 15q^2 + 30q^3 + 45q^4 + 50q^5 + 50q^6 + 50q^7 + 50q^8 + \cdots \]
\[ f_6(q) = 1 + 6q + 21q^2 + 50q^3 + 90q^4 + 126q^5 + 146q^6 + 150q^7 + 156q^8 + 152q^9 + 156q^{10} + 150q^{11} + 158q^{12} + 150q^{13} + 156q^{14} + 152q^{15} + 156q^{16} + 150q^{17} + 158q^{18} + \cdots \]
\[ f_7(q) = 1 + 7q + 28q^2 + 77q^3 + 161q^4 + 266q^5 + 364q^6 + 427q^7 + 462q^8 + 483q^9 + 490q^{10} + 490q^{11} + 490q^{12} + 490q^{13} + \cdots \]

**Notice:**
- The coefficients eventually repeat.

**Goals:**
- Find a formula for the generating function \( f_n(q) \).
- Understand this periodicity.
Pattern Avoidance Characterization

*Key idea:* (Green, 2002)

\[ \tilde{w} \text{ is fully commutative} \iff \tilde{w} \text{ is 321-avoiding}. \]

*Example.* \([-4, -1, 1, 14]\) is **NOT** fully commutative because:

\[
\begin{array}{cccccccccc}
\vdots & w(-4) & w(-3) & w(-2) & w(-1) & w(0) & w(1) & w(2) & w(3) & w(4) & \vdots \\
\tilde{w} & \vdots & 6 & -8 & -5 & -3 & 10 & -4 & -1 & 14 & \vdots \\
\end{array}
\]

\[
\begin{array}{cccccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\tilde{w} & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array}
\]
Game plan

**Goal:** Enumerate 321-avoiding affine permutations $\tilde{w}$.

- Write $\tilde{w} = w^0 w$, where $w^0 \in \tilde{S}_n / S_n$ and $w \in S_n$.
  - $w^0$ determines the entries; $w$ determines their order.

**Example.** For $\tilde{w} = [-11, 20, -3, 4, 11, 0] \in \tilde{S}_6$,

$$w^0 = [-11, -3, 0, 4, 11, 20] \text{ and } w = [1, 3, 6, 4, 5, 2].$$

- Determine which $w^0$ are 321-avoiding.
- Determine the finite $w$ such that $w^0 w$ is still 321-avoiding.
Normalized abacus and 321-avoiding criterion for $\tilde{S}_n/S_n$

We use a *normalized* abacus diagram; shifts all beads so that the first gap is in position $n+1$; this map is invertible.

**Theorem.** (H–J ‘09) Given a normalized abacus for $w^0 \in \tilde{S}_n/S_n$, where the last bead occurs in position $i$,

$$w^0 \text{ is fully commutative} \iff \text{lowest beads in runners only occur in } \{1, \ldots, n\} \cup \{i-n+1, \ldots, i\}$$

**Idea:** Lowest beads in runners $\leftrightarrow$ entries in base window.

<table>
<thead>
<tr>
<th>$w(-n+1)$ $w(-n+2)$ $\ldots$ $w(-1)$ $w(0)$</th>
<th>$w(1)$ $w(2)$ $\ldots$ $w(n-1)$ $w(n)$</th>
<th>$w(n+1)$ $w(n+2)$ $\ldots$ $w(2n-1)$ $w(2n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>lo lo $\ldots$ hi hi</td>
<td>lo lo $\ldots$ hi hi</td>
<td>lo lo $\ldots$ hi hi</td>
</tr>
<tr>
<td>lo lo med hi $\mathbf{hi}$</td>
<td>lo lo $\mathbf{med}$ hi hi</td>
<td>lo lo med hi hi</td>
</tr>
</tbody>
</table>
Long versus short elements

Partition $\widetilde{S}_n$ into long and short elements:

**Short elements**
- Lowest bead in position $i \leq 2n$
- Finitely many
- **Hard to count**

**Long elements**
- Lowest bead in position $i > 2n$
- Come in infinite families
- **Easy to count**
- *Explain the periodicity*
Enumerating long elements

For long elements $\tilde{w} \in \tilde{S}_n$, the base window for $w^0$ is $[a, a, \ldots, a, b, b, \ldots, b]$ where $1 \leq a \leq n$, and $n + 2 \leq b$.

**Question:** Which permutations $w \in S_n$ can be multiplied into a $w^0$?

- We can not invert any pairs of $a$’s, nor any pairs of $b$’s. (Would create a 321-pattern with an adjacent window)
- Only possible to *intersperse* the $a$’s and the $b$’s.

How many ways to intersperse $(k)$ $a$’s and $(n - k)$ $b$’s? \[ (\begin{array}{c} n \\ k \end{array}) \]

**BUT:** We must also keep track of the *length* of these permutations. This is counted by the $q$-binomial coefficient:

\[ \left[ \begin{array}{c} n \\ k \end{array} \right]_q = \frac{(q)_n}{(q)_k(q)_{n-k}}, \text{ where } q_n = (1 - q)(1 - q^2) \cdots (1 - q^n) \]
Enumerating long elements

After we:

- Enumerate by length all possible $w^0$ with $(k)$ $a$’s and $(n - k)$ $b$’s.
- Combine the Coxeter lengths by $\ell(\tilde{w}) = \ell(w^0) + \ell(w)$.

Then we get:

**Theorem.** (H–J ’09) For a fixed $n \geq 0$, the generating function by length for long fully commutative elements $\tilde{w} \in \tilde{S}^\text{FC}_n$ is

$$\sum q^{\ell(\tilde{w})} = \frac{q^n}{1 - q^n} \sum_{k=1}^{n-1} \binom{n}{k}^2 q^k.$$
Periodicity of fully commutative elements in $\tilde{S}_n$

**Corollary.** (H–J ’09) The coefficients of $f_n(q)$ are eventually periodic with period dividing $n$.

When $n$ is prime, the period is 1:

$$a_i = \frac{1}{n} \left( \binom{2n}{n} - 2 \right).$$

**Proof.** For $i$ sufficiently large, all elements of length $i$ are long. Our generating function is simply some polynomial over $(1 - q^n)$:

$$\frac{q^n}{1 - q^n} \sum_{k=1}^{n-1} \left[ \begin{array}{c} n \\ k \end{array} \right]_q^2 = \frac{P(q)}{1 - q^n} = P(q)(1 + q^n + q^{2n} + \cdots)$$

When $n$ is prime, an extra factor of $(1 + q + \cdots + q^{n-1})$ cancels:

$$\frac{1}{1 - q} \left[ \frac{q^n}{1 + q + \cdots + q^{n-1}} \sum_{k=1}^{n-1} \left[ \begin{array}{c} n \\ k \end{array} \right]_q^2 \right]$$

As suggested by a referee, we know that $a_i = P(1) = \frac{1}{n} \sum_{k=1}^{n-1} \binom{n}{k}^2$. 
Short elements are hard

For short elements \( \tilde{w} \in \tilde{S}_n \), the base window for \( w^0 \) is \( [a, \ldots, a, b, \ldots, b, c, \ldots, c] \), and there is more interaction:

No \( a \) can invert with an \( a \) or \( b \). No \( c \) can invert with a \( b \) or \( c \). 

- Count \( \tilde{w} \) where some \( a \) intertwines with some \( c \).
- Count \( \tilde{w} \) w/o intertwining and 0 descents in the \( b \)'s.
- Count \( \tilde{w} \) w/o intertwining and 1 descent in the \( b \)'s.
  - Not so hard to determine the acceptable finite permutations \( w \).
  - Such as \( \sum_{M \geq 0} x^{L+M+R} \sum_{\mu=1}^{M-1} \left( \begin{bmatrix} M \\ \mu \end{bmatrix} q - 1 \right) \begin{bmatrix} L+\mu \\ \mu \end{bmatrix} q^{[R+M-\mu]} \right)_q \)
- Count \( \tilde{w} \) w/o intertwining and 2 descents in the \( b \)'s.
- Count \( \tilde{w} \) which are finite permutations. (Barcucci et al.)
  - Solve functional recurrences (Bousquet-Mélou)
  - Such as
    \[
    D(x, q, z, s) = N(x, q, z, s) + \frac{xqs}{1 - qs} \left( D(x, q, z, 1) - D(x, q, z, qs) \right) + xsD(x, q, z, s)
    \]
Future Work

- Extend to $\tilde{B}_n$, $\tilde{C}_n$, and $\tilde{D}_n$
  - Develop combinatorial interpretations ✓
  - 321-avoiding characterization?
- Heap interpretation of fully commutative elements
  - Can use Viennot's heaps of pieces theory
  - Better bound on periodicity
- More combinatorial interpretations for $\tilde{W}/W$
  - What do you know?
Combinatorial interpretations of $\tilde{S}_n/S_n$

[-4, -3, 7, 10]

window notation

reduced expression

abacus diagram

bounded partition

core partition

root lattice point

$(-1, 2, 1, -2)$
Combinatorial interpretations of $\tilde{C}/C$, $\tilde{B}/B$, $\tilde{B}/D$, $\tilde{D}/D$
Future Work

- Extend to $\widetilde{B}_n$, $\widetilde{C}_n$, and $\widetilde{D}_n$
  - Develop combinatorial interpretations ✓
  - 321-avoiding characterization?
- Heap interpretation of fully commutative elements
  - Can use Viennot's heaps of pieces theory
  - Better bound on periodicity
- More combinatorial interpretations for $\widetilde{W}/W$
  - What do you know?
Thank you!

Slides available: people.qc.cuny.edu/chanusa > Talks


Christopher R. H. Hanusa and Brant C. Jones. Abacus models for parabolic quotients of affine Coxeter groups