A quasi-polynomial q-Queens result and related Kronecker products of matrices

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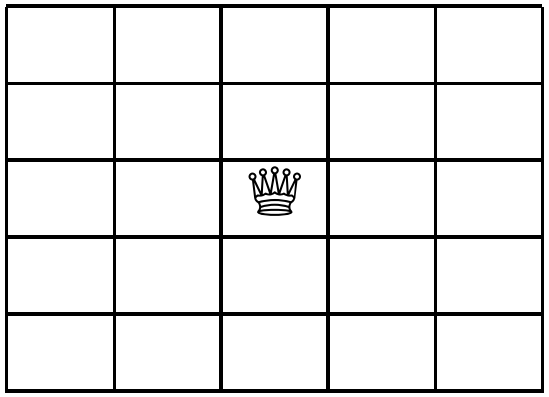
Outline

- The q-Queens Problem
- Inside-out polytopes
- Calculating quasipolynomial periods
- Kronecker products of matrices
- Open directions

Joint work with
Thomas Zaslavsky, Binghamton University
and Seth Chaiken, University at Albany.

The n-Queens Problem

Motivating question:
Can you place n nonattacking queens on an $n \times n$ chessboard?



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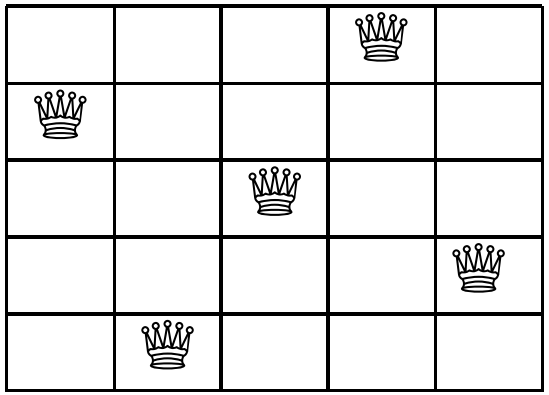

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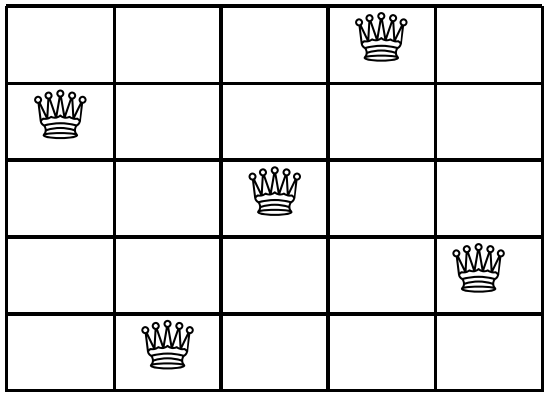

Q: In how many ways can you place n nonattacking queens?

n	1	2	3	4	5	6	7	8	9	10
#	1	0	0	2	10	4	40	92	352	724

The q-Queens Problem and generalizations

Let's generalize.

- Fix the number of queens. (q)
- Let the size of the board vary. ($n \times n$)

Q: Determine the number of ways in which you can place q nonattacking queens on an $n \times n$ chessboard as a function of n.

Q: Why stop there?

The q-Queens Problem and generalizations

A problem will have three elements:

- A piece. (A set of basic moves.)
- A board. (A convex polygon and its dilations.)
- A number. (A number of pieces to arrange.)

A **piece** P moves from $z = (x, y)$ to $z + \alpha m_r$ for $m_r \in \mathbf{M}$, $\alpha \in \mathbb{Z}$

Two pieces in z_i and z_j are attacking if $z_i - z_j = \alpha m_r$.

Examples:

- ♕ Queens: $\mathbf{M} = \{(1,0),(1,1),(0,1),(1,-1)\}$
- ♗ Bishops: $\mathbf{M} = \{(1,1),(1,-1)\}$
- ♘ Nightrider: $\mathbf{M} = \{(2,1),(1,2),(2,-1),(1,-2)\}$

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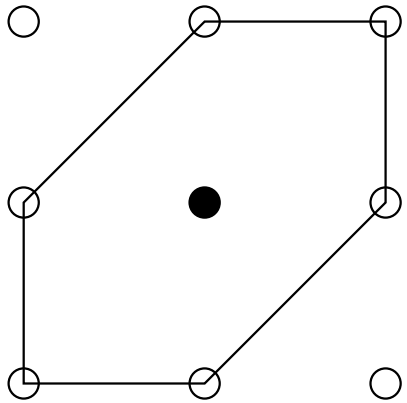
A **board** is the set of integral points on the interior of an integral multiple of a rational convex polygon $\mathcal{B} \subset \mathbb{R}^2$



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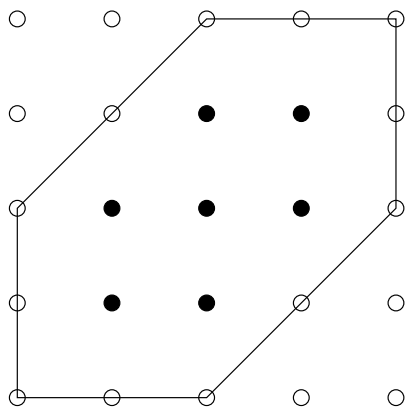

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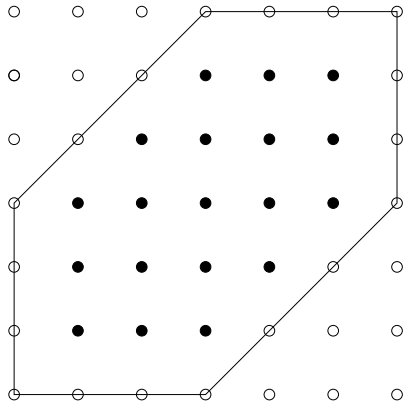


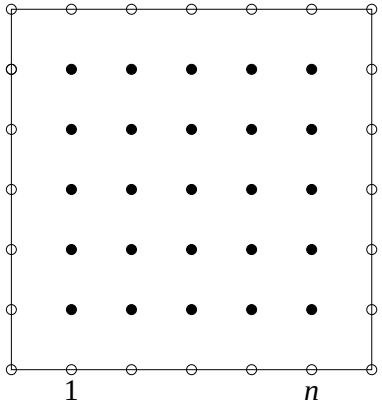
The q-Queens Problem and generalizations

Question: Given a piece P, a polygon $\mathcal{B}$, and a number q, Determine the number of ways in which you can place q nonattacking P pieces on the board $t\mathcal{B}^\circ$ as a function of t.

In the original q-Queens Problem,

- $P = \♕$
- $\mathcal{B} = [0,1]^2$

The $n \times n$ case corresponds to $t = (n+1)$.



The q-Queens Problem and generalizations

Theorem: (Chaiken, Zaslavsky, 2005)
Given P, q, and $\mathcal{B}$, the number of placements of q nonattacking P pieces inside $t\mathcal{B}$ is a quasipolynomial function of t.

A *quasipolynomial* is a function $f(t)$ on $t \in \mathbb{Z}_+$ such that

$$f(t) = c_d t^d + c_{d-1} t^{d-1} + \cdots + c_0,$$

where each c_i is periodic.

Example: $f(t) = \begin{cases} t^2 + 3t + 2 & \text{for even } t \\ t^2 - 2t + 1 & \text{for odd } t \end{cases}$

n-Queens proof sketch

Very briefly:

The rules of nonattack correspond to forbidden hyperplanes in $\mathbb{R}^{2q}$.

Inside-out polytope theory gives a quasipolynomial function of t.

n-Queens proof sketch

Less briefly:

- Goal: Count allowed unordered configuration of pieces.
- Instead, count allowed *ordered* configurations of $z_i = (x_i, y_i)$.

- A configuration is a point $(x_1, y_1, \ldots, x_q, y_q) \in \mathbb{Z}^{2q} \cap t\mathcal{B}^q$
- Two pieces are attacking when $(z_j - z_i) \cdot m_r^{\perp} = 0$.
- There are $\binom{q}{2} N$ of these forbidden hyperplanes in $\mathbb{R}^{2q}$

- Count lattice points in $t\mathcal{B}^q$ avoiding $\mathcal{H}$.
- Inside-out polytope theory yields a quasipolynomial with certain properties.

Inside-out polytopes

Introduced by Beck and Zaslavsky, 2006, building upon ideas of Ehrhart theory, an *inside-out polytope* $(\mathcal{P}, \mathcal{H})$ is a convex polytope $\mathcal{P}$ together with an arrangement of hyperplanes, $\mathcal{H}$, dissecting $\mathcal{P}$.

- Counts t^{-1}-fractional points inside $\mathcal{P}$.
- Requires that the vertices of $\mathcal{P}$ are in fractional lattice and the $\mathcal{H}$ have rational equations.
- The $\mathcal{H}$ are homogeneous.

integral points inside $t\mathcal{B}^q$	$\longleftrightarrow$	t^{-1}-fractional points inside $\mathcal{B}^q$

Inside-out polytopes

Conclusion: The number of lattice points inside $\mathcal{P}$ avoiding $\mathcal{H}$ is a quasipolynomial function of t with

- degree: $\dim(\mathcal{P})$.
- leading coefficient: volume of $\mathcal{P}$.

Inside-out polytopes

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Therefore: The number nonattacking configurations of q pieces P inside $t\mathcal{B}$ is a quasipolynomial function of t with

- degree: $\dim(\mathcal{B}^q) = 2q$.
- leading coefficient: $|\mathcal{B}|^q/q!$. $\longrightarrow$ Now unordered!

But what does this mean?

So, we have a solution to q-Queens and n-Queens?

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So, we have a solution to q-Queens and n-Queens?

- No. The theorem only proves existence.

We must determine the periodic coefficients c_i.

Game plan:

- Determine the period of the coefficients.
- Compute initial data to determine the formula.

Rooks and bishops

Notation: Write $N_P(q; n)$ for the number of nonattacking configurations of q pieces P on an $n \times n$ board.

Recall: $\mathcal{B} = [0, 1]^2$ and $t = n + 1$, implying:

- degree of $N_P(q; n)$ is $2q$.
- leading coefficient of $N_P(q; n)$ is $1/q!$.

For a fixed q, we expect a formula of the form:

$$N_P(q; n) = \frac{1}{q!} n^{2q} + c_{2q-1} n^{2q-1} + \cdots + c_1 n + c_0$$

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Classic result for rooks R:

♖ $N_R(q;n) = q! \binom{n}{q}^2$.

Rooks and bishops

- For bishops B:
- We will calculate that $p|2^{q-1}$. (stay tuned!)

- One ♗: $N_B(1;n) = n^2$.
- Two ♗: quasipolynomial of degree 4, period 1 or 2.
- $N_B(2;n) = \frac{1}{2}n^4 + c_3 n^3 + c_2 n^2 + c_1 n + c_0$

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- Initial data for $N_B(2;n)$:

n	1	2	3	4	5	6	7	8
$N_B(2;n)$	0	4	26	92	240	520	994	1736

$$N_B(2;n) = \frac{1}{2}n^4 - \frac{2}{3}n^3 + \frac{1}{2}n^2 - \frac{1}{3}n.$$

The theory of the quasipolynomial period

Known: The period of the quasipolynomial divides the least common denominator of coordinates of *all points of intersection* of k forbidden hyperplanes with a k-dimensional face of $\mathcal{B}^q$.

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Create a matrix to determine incidences:

- Each move m_l generates equations; let $m^\perp = (m_{l1}, m_{l2})$. $m_{l1}x_i + m_{l2}y_i - m_{l1}x_j - m_{l2}y_j = 0$
- Write the boundaries as $a_l x + b_l y \leq \beta_l$

The theory of the quasipolynomial period

- Each move m_l generates equations; let $m^{\perp} = (m_{l1}, m_{l2})$.
  $m_{l1}x_i + m_{l2}y_i - m_{l1}x_j - m_{l2}y_j = 0$

$$F := \begin{pmatrix} M & -M & 0 & 0 & \cdots & 0 & 0 \\ M & 0 & -M & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ M & 0 & 0 & 0 & \cdots & 0 & -M \\ 0 & M & -M & 0 & \cdots & 0 & 0 \\ 0 & M & 0 & -M & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & M & 0 & 0 & \cdots & 0 & -M \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & M & -M \end{pmatrix} \qquad M := \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{N2} \\ \cdots & \cdots \\ m_{N1} & m_{N2} \end{pmatrix}$$

The theory of the quasipolynomial period

- Write the boundaries as $a_l x + b_l y \leq \beta_l$

$$G := \begin{pmatrix} B & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & B & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & B \end{pmatrix} \quad B := \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \cdots & \cdots \\ a_K & b_K \end{pmatrix} \quad \beta := \begin{pmatrix} \beta_1 \\ \beta_2 \\ \cdots \\ \beta_K \end{pmatrix}$$

- For a square or rectangular grid, $B := \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{matrix} -x \leq 0 \\ -y \leq 0 \\ x \leq 1 \\ y \leq 1 \end{matrix}$

The theory of the quasipolynomial period

Define $A = \begin{pmatrix} F \\ G \end{pmatrix}$ and $b = \begin{pmatrix} \mathbf{0} \\ \beta \end{pmatrix}$.

Then $Az = b$ determines all vertices of the inside-out polytope.

Cramer's Rule: Every denominator divides a $2q \times 2q$ subdeterminant of A.

Thus, the period of the counting quasipolynomial is a divisor of the least common multiple of all such subdeterminants, $\mathrm{lcmd}(A)$.

This operation $\mathrm{lcmd}(A)$ appears unknown.

Calculating lcmd(A)

We restrict to square and rectangular grids.

- The rows of G contain one non-zero entry.
- Any subdeterminant of A including a row of G can be reduced to a subdeterminant of F.
- Therefore, $\mathrm{lcmd}(A) = \mathrm{lcmd}(F) = \mathrm{lcmd}(F^T)$.

The structure of F^T is particular.

The structure of F

$$F := \begin{pmatrix} M & -M & 0 & 0 & \cdots & 0 & 0 \\ M & 0 & -M & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ M & 0 & 0 & 0 & \cdots & 0 & -M \\ 0 & M & -M & 0 & \cdots & 0 & 0 \\ 0 & M & 0 & -M & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & M & 0 & 0 & \cdots & 0 & -M \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & M & -M \end{pmatrix} \qquad M := \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{N2} \\ \cdots & \cdots \\ m_{N1} & m_{N2} \end{pmatrix}$$

The structure of F^T

$$F^T = \begin{pmatrix} M^T & M^T & \cdots & M^T & 0 & 0 & \cdots & 0 & \cdots & 0 \\ -M^T & 0 & \cdots & 0 & M^T & M^T & \cdots & M^T & \cdots & 0 \\ 0 & -M^T & \cdots & 0 & -M^T & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & -M^T & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & M^T \\ 0 & 0 & \cdots & -M^T & 0 & 0 & \cdots & -M^T & \cdots & -M^T \end{pmatrix}$$

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This reminds us of the incidence matrix for the graph K_q:

$$D(K_q) = \begin{pmatrix} 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & \cdots & 0 \\ -1 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 & \cdots & 0 \\ 0 & -1 & \cdots & 0 & -1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & -1 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 \\ 0 & 0 & \cdots & -1 & 0 & 0 & \cdots & -1 & \cdots & -1 \end{pmatrix}$$

The structure of F^T

For matrices $A = (a_{ij})_{m \times k}$ and $B = (b_{ij})_{n \times l}$, the *Kronecker product* $A \otimes B$ is defined to be the $mn \times kl$ block matrix

$$\left[\begin{array}{c:c:c} a_{11}B & \cdots & a_{1k}B \\ \hdashline \vdots & \ddots & \vdots \\ \hdashline a_{m1}B & \cdots & a_{mk}B \end{array}\right].$$

- $F^T = D(K_q) \otimes M^T$,
- M is the $m \times 2$ moves matrix
- $D(K_q)$ is the incidence matrix for the complete graph K_q.

About Kronecker Products

About Kronecker products:

- $A \otimes B$ and $B \otimes A$ only differ by row and column switchings.
- For $A_{m \times m}$ and $B_{n \times n}$, $\det(A \otimes B) = \det(A)^n \det(B)^m$.

- Calculating $\mathrm{lcmd}(A \otimes B)$ appears difficult for generic A, B.
- We aim to simplify $\mathrm{lcmd}(M^T \otimes D(K_q))$.
- Funny story.

lcmd result

Theorem (Hanusa, Zaslavsky, 2008) Given $M_{m\times 2}$ and $q \geq 1$,

$$\operatorname{lcmd}\left(M \otimes D(K_q)\right) = \operatorname{lcm}\left((\operatorname{lcmd} M)^{q-1}, \operatorname*{LCM}_{\mathcal{K}}\left(\prod_{(I,J)\in\mathcal{K}} \det M^{I,J}\right)\right),$$

The LCM is over disjoint multisubsets I and J of $[m]$ of size $\lfloor q/2 \rfloor$...

- **This reduces the** lcmd **to an** lcm **over entries of** M**.**

lcmd result

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The LCM is over disjoint multisubsets I and J of $[m]$ of size $\lfloor q/2 \rfloor$,

$$\text{and } M^{I,J} = \begin{pmatrix} \prod m_{i1} & \prod m_{i2} \\ \prod m_{j1} & \prod m_{j2} \end{pmatrix}$$

Example: For $M_{4\times 2}$, we have $m = 4$.

Find all pairs (I, J) of disjoint $\lfloor q/2 \rfloor$-member multisubsets of $[4]$:

$$(\{1^a\}, \{2^b, 3^c, 4^d\}), \quad (\{2^b\}, \{1^a, 3^c, 4^d\}),$$
$$(\{3^c\}, \{1^a, 2^b, 4^d\}), \quad (\{4^d\}, \{1^a, 2^b, 3^c\}),$$
$$(\{1^a, 2^b\}, \{3^c, 4^d\}), \quad (\{1^a, 3^c\}, \{2^b, 4^d\}), \quad (\{1^a, 4^d\}, \{2^b, 3^c\}).$$

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Example: For $M_{4\times 2}$, $m = 4$. Let $(I, J) = (\{1^a, 2^b\}, \{3^c, 4^d\})$.

$$\text{Then, } M^{I,J} = \begin{pmatrix} m_{11}^a m_{21}^b & m_{12}^a m_{22}^b \\ m_{31}^c m_{41}^d & m_{32}^c m_{42}^d \end{pmatrix},$$

For all a, b, c, and d such that $a + b = c + d = \lfloor q/2 \rfloor$.

Bishop example

Example: When $P =$ ♗ (bishop),

$$M = M^T = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

It is clear that $\operatorname{lcmd}(M) = 2$. Applying the theorem,

$$\operatorname{lcmd}\left(M \otimes D(K_q)\right) = \operatorname{lcm}\left(2^{q-1}, \operatorname*{LCM}_{p=2}^{\lfloor q/2 \rfloor} \left((-1)^p - 1^p\right)^{\lfloor q/2p \rfloor}\right).$$

The LCM term generates powers of 2 no larger than $2^{q/2}$.

Hence, $\operatorname{lcmd}\left(M \otimes D(K_q)\right) = 2^{q-1}$.

Back to that funny story...

Sketch of Kronecker theorem proof

$$\operatorname{lcmd}\left(M \otimes D(K_q)\right) = \operatorname{lcm}\left((\operatorname{lcmd} M)^{q-1}, \operatorname*{LCM}_{\mathcal{K}}\left(\prod_{(I,J)\in\mathcal{K}} \det M^{I,J}\right)\right),$$

Goal: Show that every $N_{l\times l}$ subdet. of $M \otimes D(K_q)$ divides RHS.

- Consider only N such that $\det(N) \neq 0$.
- N is a choice of l rows and l columns from $M \otimes D(K_q)$
- Same as a choice of l vertices and l edges from K_q, with up to m copies of each vertex and up to two copies of an edge.

$$M \otimes D(K_q) = \begin{pmatrix} m_{11}D(K_q) & m_{12}D(K_q) \\ m_{21}D(K_q) & m_{22}D(K_q) \\ \vdots & \vdots \\ m_{m1}D(K_q) & m_{m2}D(K_q) \end{pmatrix}$$

Sketch of Kronecker theorem proof

- When two rows correspond to the same vertex v, the rows contain the same entries except for different multipliers m_{ik}.

$$\begin{pmatrix} m_{11}D(K_q) & m_{12}D(K_q) \\ m_{21}D(K_q) & m_{22}D(K_q) \\ \vdots & \vdots \\ m_{m1}D(K_q) & m_{m2}D(K_q) \end{pmatrix} \qquad \begin{pmatrix} \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ m_{i1} & -m_{i1} & 0 & \cdots & 0 & m_{i2} & 0 & \cdots & -m_{i2} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ m_{j1} & -m_{j1} & 0 & \cdots & 0 & m_{j2} & 0 & \cdots & -m_{j2} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \end{pmatrix}$$

- A vertex chosen three or more times would imply lin. dep.
- Simplify $\det N$ when a vertex is chosen twice. (This generates a factor of $\det M^{i,j}$).

$$\begin{pmatrix} \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ m_{i1} & -m_{i1} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & m_{j2} & 0 & \cdots & -m_{j2} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \end{pmatrix}$$

Sketch of Kronecker theorem proof

- Afterwards, every column has at most two entries.
- If a row (or column) has exactly one non-zero entry, expand.
- Then every row has exactly two non-zero entries as well.
- This matrix breaks down as a product of incidence matrices of weighted cycles, each of which basically contributes $\det M^{I,J}$.

$$\begin{pmatrix} y_1 & 0 & 0 & 0 & 0 & -z_6 \\ -z_1 & y_2 & 0 & 0 & 0 & 0 \\ 0 & -z_2 & y_3 & 0 & 0 & 0 \\ 0 & 0 & -z_3 & y_4 & 0 & 0 \\ 0 & 0 & 0 & -z_4 & y_5 & 0 \\ 0 & 0 & 0 & 0 & -z_5 & y_6 \end{pmatrix}.$$

Applying the Kronecker theorem, I

When $M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & -1 \end{pmatrix}$, again $\operatorname{lcmd}(M) = 2$.

We then calculate $\det M^{I,J}$ for each pair (I, J).
For example, when $I = \{3^c\}$ and $J = \{1^a, 2^b, 4^d\}$, and

$$M^{I,J} = \begin{pmatrix} 1^c & 1^c \\ 1^a 0^b 1^d & 0^a 1^b (-1)^d \end{pmatrix}.$$

where $c = a + b + d = \lfloor q/2 \rfloor$.

Applying the Kronecker theorem, I

When $M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & -1 \end{pmatrix}$, again $\operatorname{lcmd}(M) = 2$.

We then calculate $\det M^{I,J}$ for each pair (I, J).
For example, when $I = \{3^c\}$ and $J = \{1^a, 2^b, 4^d\}$, and

$$M^{I,J} = \begin{pmatrix} 1^c & 1^c \\ 1^a 0^b 1^d & 0^a 1^b (-1)^d \end{pmatrix}.$$

where $c = a + b + d = \lfloor q/2 \rfloor$.

- The only non-trivial case is when $a = b = 0$. Therefore $c = d = \lfloor q/2 \rfloor$ and $\det M^{I,J} = 0$ or -2.
- This implies that the LCM in the theorem divides 2^{q-1}.

We conclude that $\operatorname{lcmd}\left(M \otimes D(K_q)\right) = 2^{q-1}$.

Applying the Kronecker theorem, II

Consider $M = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 1 & -2 \\ 2 & -1 \end{pmatrix}$.

- The submatrices

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}, \text{ and } \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix},$$

  have determinants -3, -4, and -5; hence $\operatorname{lcmd}(M) = 60$.
- We have the same multisubsets of $[4]$ as before.
- $\det M^{I,J}$ is the same form in all cases: $\pm 2^u(2^{2\lfloor q/2 \rfloor - 2u} \pm 1)$, where u is a number between 0 and $\lfloor q/2 \rfloor$.

Applying the Kronecker theorem, II

We conclude that

$$\operatorname{lcmd}\left(M \otimes D(K_q)\right) = \operatorname{lcm}\left(60^{q-1}, \operatorname*{LCM}_{\substack{1 \le p \le q/2 \\ 0 \le u \le p-1}} (2^{2p-2u} \pm 1)^{\lfloor q/2p \rfloor}\right).$$

When $q = 8$, $\operatorname{lcmd}\left(M \otimes D(K_8)\right) =$

$$\operatorname{lcm}\left(60^7, (4 \pm 1)^{\lfloor 8/2 \rfloor}, (16 \pm 1)^{\lfloor 8/4 \rfloor}, (64 \pm 1)^{\lfloor 8/6 \rfloor}, (256 \pm 1)^{\lfloor 8/8 \rfloor}\right)$$

$$= 60^7 \cdot 7 \cdot 13 \cdot 17^2 \cdot 257.$$

Applying the Kronecker theorem, II

The first few values of q give the following numbers:

q	lcmd $\left(M \otimes D(K_q)\right)$	(factored)
2	60	60^1
3	3600	60^2
4	3672000	$60^3 \cdot 17$
5	220320000	$60^4 \cdot 17$
6	1202947200000	$60^5 \cdot 7 \cdot 13 \cdot 17$
7	72176832000000	$60^6 \cdot 7 \cdot 13 \cdot 17$
8	18920434740480000000	$60^7 \cdot 7 \cdot 13 \cdot 17^2 \cdot 257$
9	1135226084428800000000	$60^8 \cdot 7 \cdot 13 \cdot 17^2 \cdot 257$
10	952295753183943168000000000	$60^9 \cdot 7 \cdot 11 \cdot 13 \cdot 17^2 \cdot 31 \cdot 41 \cdot 257$

View of our wandering from above

- Generalize n-Queens to q-Queens and beyond.
- Apply inside-out polytope theory to prove formula existence.
- Need to know the period to have any chance; find $\mathrm{lcmd}(A)$.
- On a rectangular board, $\mathrm{lcmd}(A) = \mathrm{lcmd}(M^T \otimes D(K_q))$.
- Prove a theorem that applies to find $\mathrm{lcmd}(M \otimes D(K_q))$.
- The theorem applies for $M_{2\times 2}$.

Open problems

- ♕ A fix for queens is next on the plate.
- ♘ Nightriders look difficult.
- $\otimes$ What goes wrong with more than two columns?
- ¿ Is a formula too much to ask?
- $p \cdot q$ When do two multivariate binomials have a common divisor?

lcmd result

Theorem (Hanusa, Zaslavsky, 2008) Given $M_{m\times 2}$ and $q \geq 1$,

$$\operatorname{lcmd}\left(M \otimes D(K_q)\right) = \operatorname{lcm}\left((\operatorname{lcmd} M)^{q-1}, \operatorname*{LCM}_{\mathcal{K}}\left(\prod_{(I,J)\in\mathcal{K}} \det M^{I,J}\right)\right),$$

The LCM is over disjoint multisubsets I and J of $[m]$ of size $\lfloor q/2 \rfloor$...

- **This reduces the** lcmd **to an** lcm **over entries of** M**.**

Open problems

- ♕ A fix for queens is next on the plate.
- ♘ Nightriders look difficult.
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- ¿ Is a formula too much to ask?
- $p \cdot q$ When do two multivariate binomials have a common divisor?

$$(wx^2y - z^2u^2) \text{ and } (wy^3 - xz^2u)$$

Thank you

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